Exponentially Faster Shortest Paths in the Congested Clique

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This project has received funding from the European Union's Horizon 2020
Research and Innovation Programme under grant agreement no. 755839

This Talk

$poly(\log \log n)$ round algorithms for approximate shortest paths in the Congested Clique

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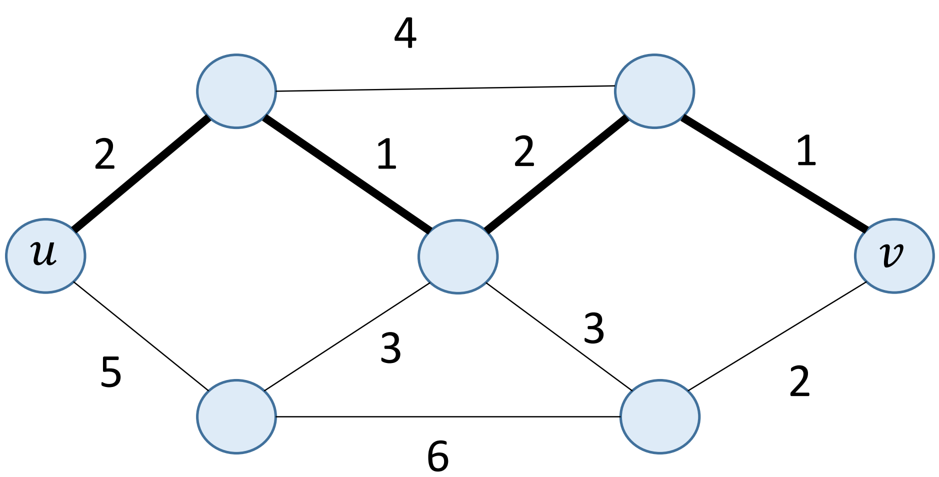
- Background & our results
- Techniques:
  - Near-additive emulators
  - Distance sensitive toolkit

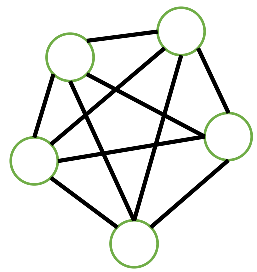
Background & Our Results

Distance Computation

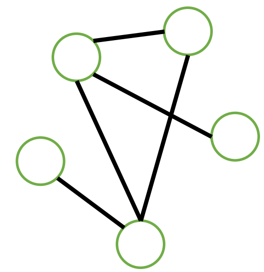
- All-pairs shortest paths (APSP)
- Single-source shortest paths (SSSP)
- Multi-source shortest paths (MSSP)



The Congested Clique Model



Communication Network



Input Graph

- n vertices
- Synchronous rounds, $\Theta(\log n)$-bit messages
- All-to-All communication
- Input and output are local

Previous Work

- Polynomial time algorithms for exact APSP based on matrix multiplication [Censor-Hillel et al. 15, Le Gall 16]

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Round Complexity	Variant
$\tilde{O}(n^{1/3})$	weighted directed
$O(n^{0.158})$	unweighted undirected

Previous Work

- Polynomial time algorithms for exact APSP based on matrix multiplication [Censor-Hillel et al. 15, Le Gall 16]

- Poly-logarithmic algorithms for approximate shortest paths [Becker et al. 17, Censor-Hillel et al. 19]

Previous Work: Approximations

Round Complexity	Problem	Reference
$O(\epsilon^{-3}\text{polylog}\, n)$	$(1+\epsilon)$-**SSSP**	[Becker, Karrenbauer, Krinninger, Lenzen '17]
$O(\log^2 n/\epsilon)$	$(1+\epsilon)$-**MSSP** with $O(\sqrt{n})$ sources	[Censor-Hillel, D, Korhonen, Leitersdorf, '19]
$O(\log^2 n/\epsilon)$	$(3+\epsilon)$-**APSP**	
$O(\log^2 n/\epsilon)$	$(2+\epsilon)$-**unweighted APSP**	

- All results are for **weighted undirected** graphs, unless specified otherwise

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Distances in the Clique

Can we get faster algorithms?

- In the matrix multiplication algorithms, in iteration i we deal with paths of 2^i edges
- We need $\log n$ iterations

Our Results

Unweighted undirected graphs:

$poly(\log\log n)$	• $(1+\epsilon)$-**MSSP** with $O(n^{1/2})$ sources • $(2+\epsilon)$-**APSP**

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Our Results

Unweighted undirected graphs:

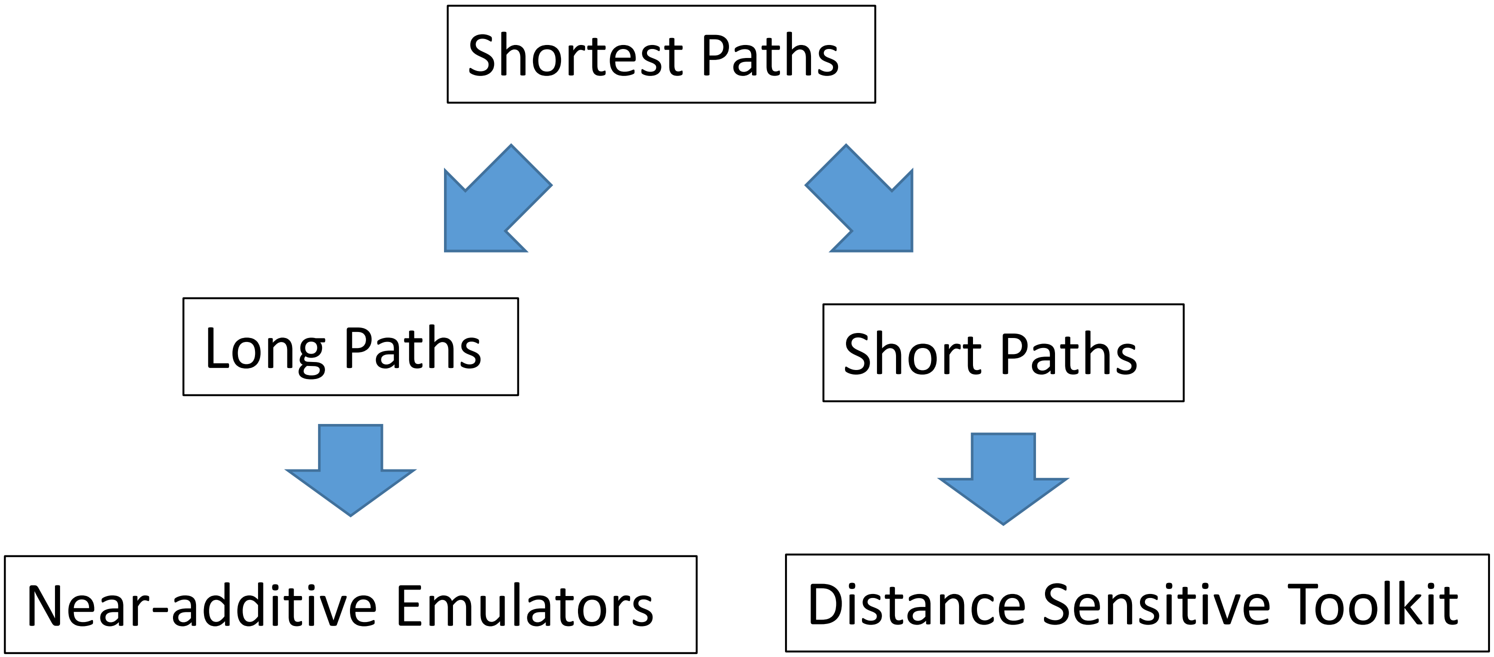
$poly(\log\log n)$	• $(1+\epsilon)$-**MSSP** with $O(n^{1/2})$ sources • $(2+\epsilon)$-**APSP**

The APSP approximation is near-tight:

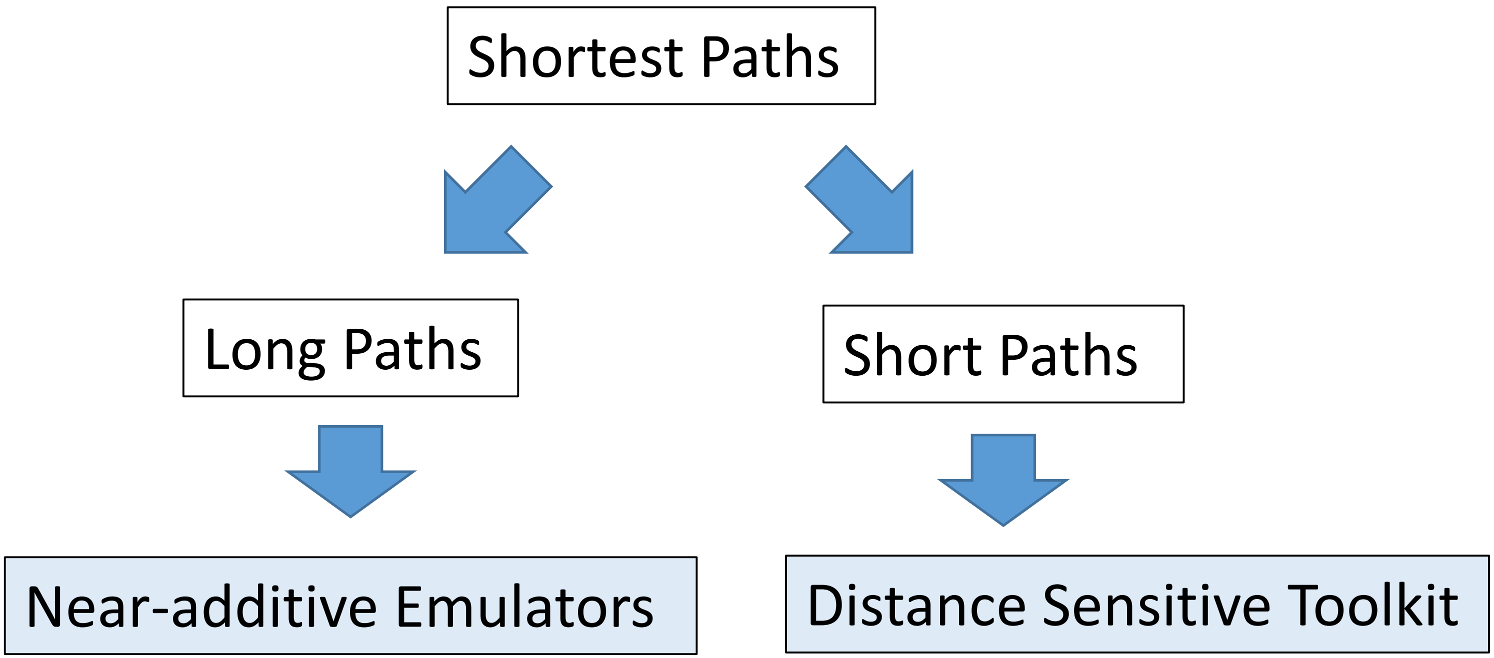
$(2-\epsilon)$-APSP implies Matrix Multiplication	[Dor, Halperin, Zwick '00 Korhonen, Suomela '18]

Techniques

Our Techniques



Our Techniques



How to Get Faster Algorithms?

- $poly(\log n)$ looks like a natural barrier

How to Get Faster Algorithms?

- $poly(\log n)$ looks like a natural barrier
- But maybe we can improve the approximation?

APSP Approximation

- $(2+\epsilon)$-approximation for APSP:

$$\delta(u,v) \leq (2+\epsilon)d(u,v)$$

APSP Approximation

- $(2 + \epsilon)$-approximation for APSP:

$$\delta(u, v) \le (2 + \epsilon) d(u, v)$$

- $(1 + \epsilon, \beta)$-approximation:

$$\delta(u, v) \le (1 + \epsilon) d(u, v) + \beta$$

APSP Approximation

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- $(1+\epsilon,\beta)$-approximation:

$$\delta(u,v) \leq (1+\epsilon)d(u,v)+\beta$$

Much better for long distances!

Near-Additive Emulator

A sparse graph H such that for all u, v:

$$d(u, v) \leq d_H(u, v) \leq (1 + \epsilon)d(u, v) + \beta$$

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Build near-additive emulator with $O(n \log\log n)$ edges

Collect it by all vertices in $O(\log\log n)$ rounds

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→

Collect it by all vertices in $O(\log\log n)$ rounds

$(1+\epsilon, \beta)$-APSP, for $\beta = O\left(\frac{\log\log n}{\epsilon}\right)^{\log\log n}$

Shortest Paths via Emulators

Near-additive emulators

$\downarrow$

$(1+\epsilon,\beta)$-APSP, $\beta = O\left(\frac{\log\log n}{\epsilon}\right)^{\log\log n}$

$\downarrow$

If $d(u,v) = \Omega(\beta/\epsilon)$: $(1+\Theta(\epsilon))$-approximation

Shortest Paths via Emulators

Near-additive emulators

$(1+\epsilon,\beta)$-APSP, $\beta = O\left(\frac{\log\log n}{\epsilon}\right)^{\log\log n}$

If $d(u,v) = \Omega(\beta/\epsilon)$: $(1+\Theta(\epsilon))$-approximation

Left with short paths of length $t = O(\beta/\epsilon)$

Shortest Paths via Emulators

Left with short paths of length $t = O(\beta/\epsilon)$

Requires $poly(\log t) = poly(\log\log n)$ time!

Shortest Paths via Emulators

Shortest Paths

Long Paths: $\Omega(\beta/\epsilon)$

Near-additive Emulators: $(1+\Theta(\epsilon))$-approximation

Short Paths: $O(\beta/\epsilon)$

Distance Sensitive Toolkit: $poly\left(\log\frac{\beta}{\epsilon}\right)$ time

Shortest Paths via Emulators

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Long Paths: $\Omega(\beta/\epsilon)$

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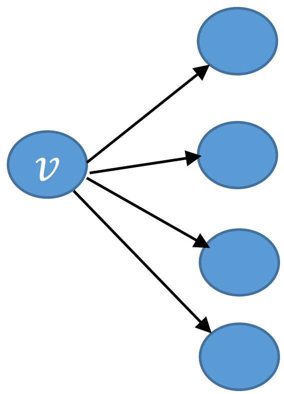
Distance Sensitive Toolkit: $poly\left(\log\frac{\beta}{\epsilon}\right)$ time

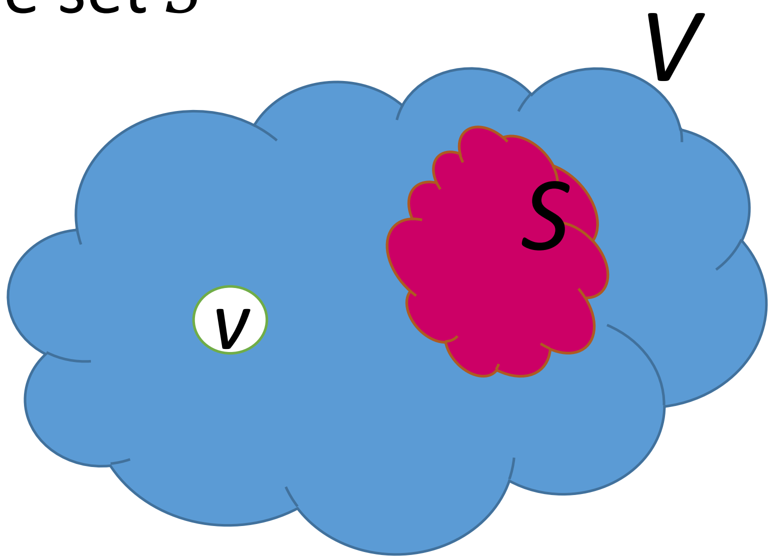
Distance Tools

[Censor-Hillel, D, Korhonen, Leitersdorf, '19]

- k-nearest: for each vertex, compute distances to k nearest vertices



- MSSP: for each vertex, compute $(1+\epsilon)$-approximate distances to the set S



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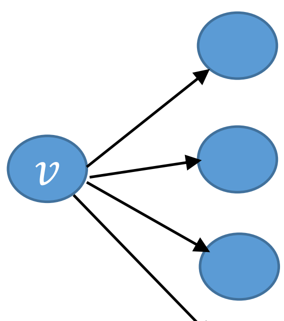

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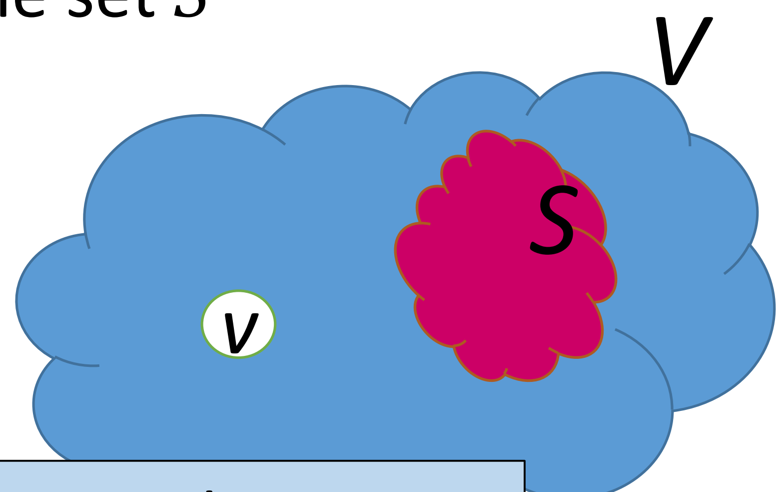
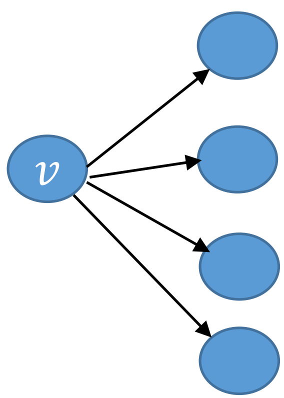


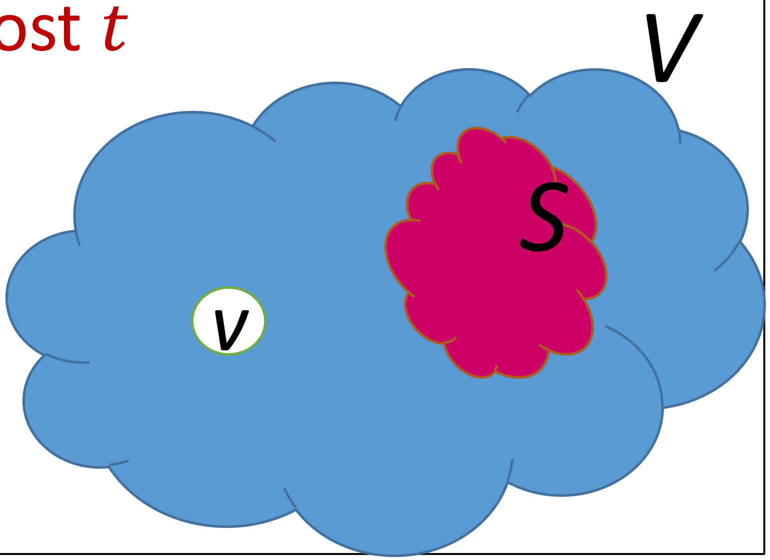
Can be implemented in $poly(\log n)$ time

Distance Sensitive Tools

- (k, t)-nearest: for each vertex, compute distances to k nearest vertices of distance at most t



- Bounded MSSP: for each vertex, compute $(1+\epsilon)$-approximate distances to vertices in S of distance at most t



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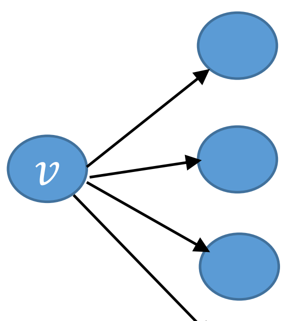

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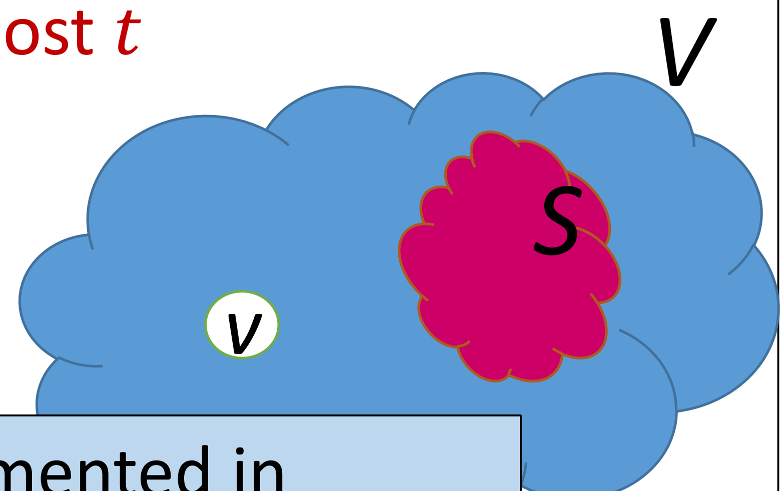

Can be implemented in $poly(\log t) = poly(\log \log n)$ time

Shortest Paths via Emulators

Shortest Paths

Long Paths: $\Omega(\beta/\epsilon)$

Short Paths: $O(\beta/\epsilon)$

Near-additive Emulators: $(1+\Theta(\epsilon))$-approximation

Distance Sensitive Toolkit: $poly\left(\log\frac{\beta}{\epsilon}\right)$ time

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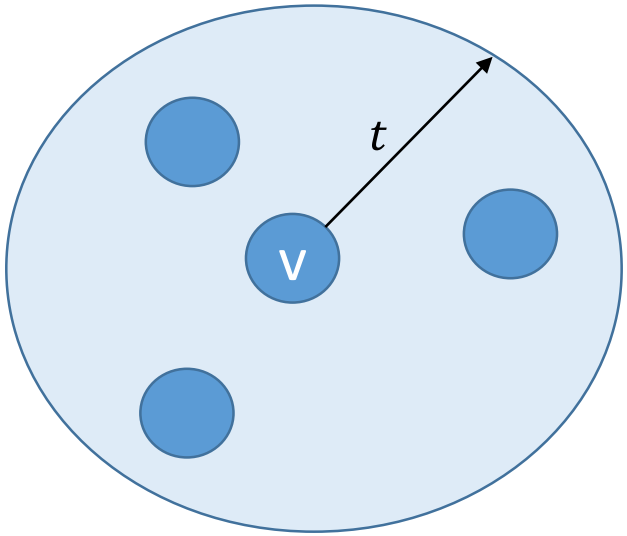
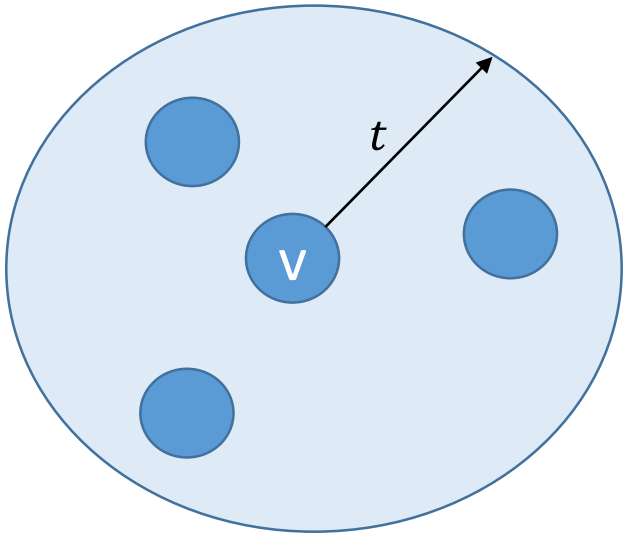
Near-Additive Emulators

- Each vertex inspects its $t = O(\beta/\epsilon)$-radius ball, and adds to the emulator edges to some of these vertices



Near-Additive Emulators

- Each vertex inspects its $t = O(\beta/\epsilon)$-radius ball, and adds to the emulator edges to some of these vertices
- Can be implemented in $poly(\log t) = poly(\log \log n)$ time using the distance sensitive toolkit



Near-Additive Emulators

Inspired by [Elkin-Neiman, 2018] and [Thorup-Zwick, 2006]

We construct:

$$\left(1+\epsilon, O\left(\frac{r}{\epsilon}\right)^{r-1}\right)\text{-emulator with } O\left(rn^{1+1/2^r}\right) \text{ edges}$$

- Choosing $r = \log\log n$ gives:

$$(1+\epsilon, \beta)\text{-emulator with } O(n \log\log n) \text{ edges,}$$

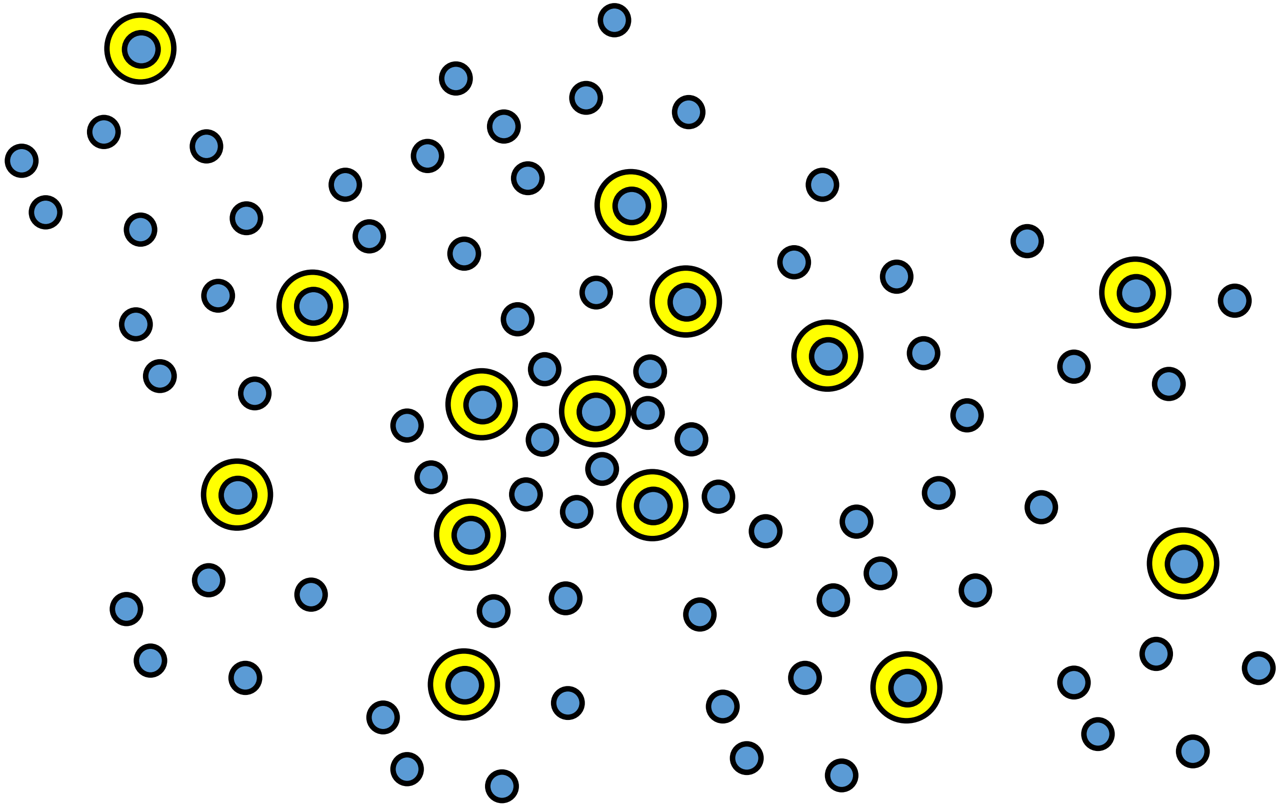
where $\beta = O\left(\frac{\log\log n}{\epsilon}\right)^{\log\log n}$

Near-Additive Emulators

- Define sampled subsets $V = S_0 \supseteq S_1 \supseteq \cdots \supseteq S_r \supseteq S_{r+1} = \emptyset$
- $S_i \leftarrow Sample(S_{i-1}, p_i)$

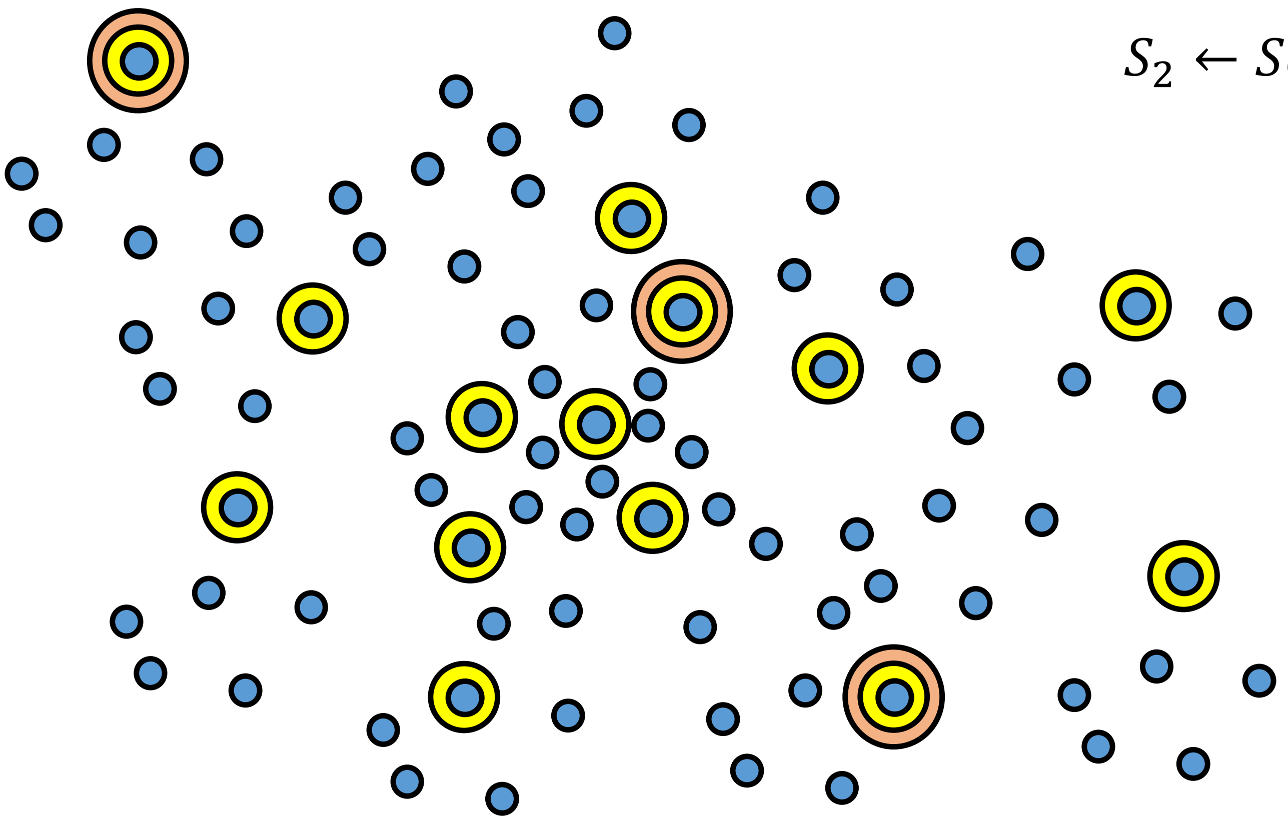
The choice of p_i determines the size of the emulator.

$$S_1 \leftarrow Sample(V, p_1)$$



$$S_1 \leftarrow Sample(V, p_1)$$

$$S_2 \leftarrow Sample(S_1, p_2)$$



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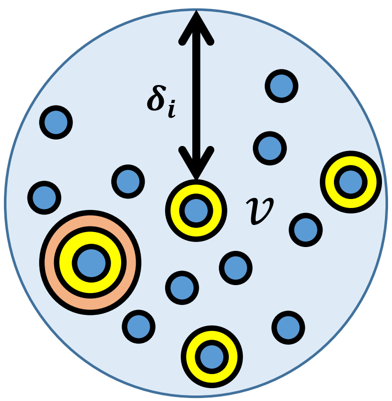
A vertex in $\boldsymbol{v} \in \boldsymbol{S_i}$ looks at the ball of radius $\boldsymbol{\delta_i} = \Theta(1/\epsilon^i)$

Is there a vertex in $\boldsymbol{B(v, \delta_i)} \cap \boldsymbol{S_{i+1}}$?



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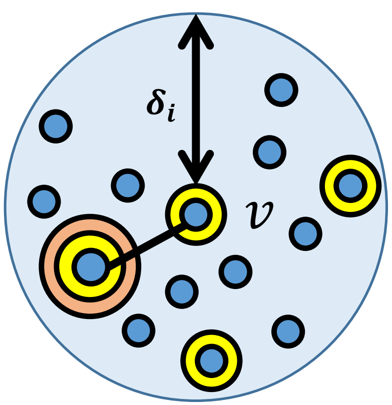

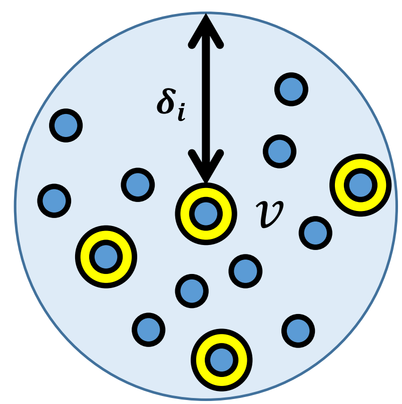

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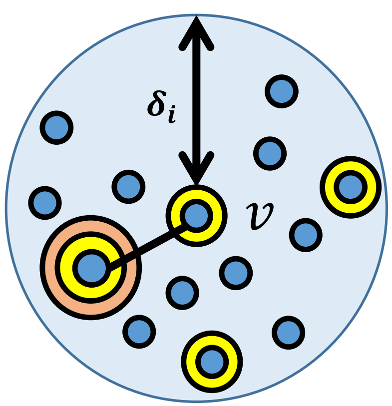

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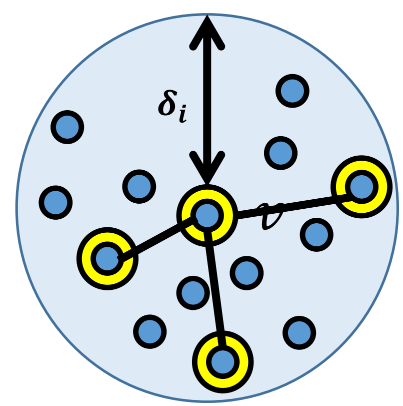
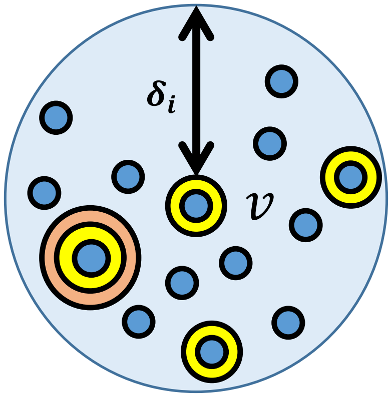


Near-Additive Emulators

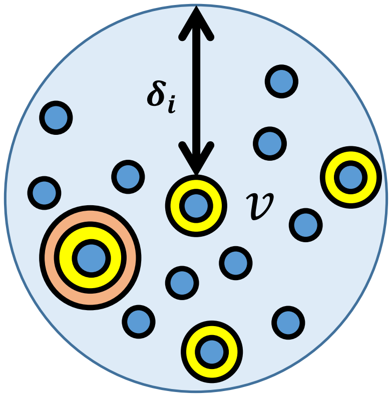
- Vertices inspect balls of radius $\boldsymbol{\delta_i}$
- Using the distance sensitive toolkit can be done in $poly(\log \delta_i)$ rounds



Stretch Analysis

- i-clustered vertex: there is a vertex in S_i close-by

Lemma: if all vertices in the shortest $u - v$ path are at most i-clustered, $d_H(u,v) \leq (1 + \Theta(\epsilon i))d(u,v) + \Theta\left(\frac{1}{\epsilon^{i-1}}\right)$



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- After rescaling: $\left(1 + \epsilon, O\left(\frac{r}{\epsilon}\right)^{r-1}\right)$

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- After rescaling: $\left(1 + \epsilon, O\left(\frac{\log\log n}{\epsilon}\right)^{\log\log n}\right)$

Shortest Paths via Emulators

Shortest Paths

Long Paths: $\Omega(\beta/\epsilon)$

Short Paths: $O(\beta/\epsilon)$

Near-additive Emulators:
$(1+\Theta(\epsilon))$-approximation

Distance Sensitive Toolkit:
$poly\left(\log\frac{\beta}{\epsilon}\right)$ time

Conclusion

Near-additive Emulators → Long Paths

Distance Sensitive Toolkit → Short Paths

$poly(\log\log n)$	$(1+\epsilon)$-**MSSP** with $O(n^{1/2})$ sources In **unweighted** graphs
$poly(\log\log n)$	$(2+\epsilon)$-**APSP** in **unweighted** graphs

Summary

$poly(\log\log n)$ round algorithms for approximate shortest paths in the Congested Clique

Unweighted graphs:

$poly(\log\log n)$	• $(1+\epsilon)$-**MSSP** with $O(n^{1/2})$ sources • $(2+\epsilon)$-**APSP** • $(1+\epsilon,\beta)$-**APSP,** $\beta = O\left(\frac{\log\log n}{\epsilon}\right)^{\log\log n}$

Open Questions

- Faster algorithms
- Weighted APSP
- Directed/exact shortest paths