Improved Distributed Approximations for Minimum-Weight Two-Edge-Connected Spanning Subgraph

Michal Dory (Technion), Mohsen Ghaffari (ETH Zurich)

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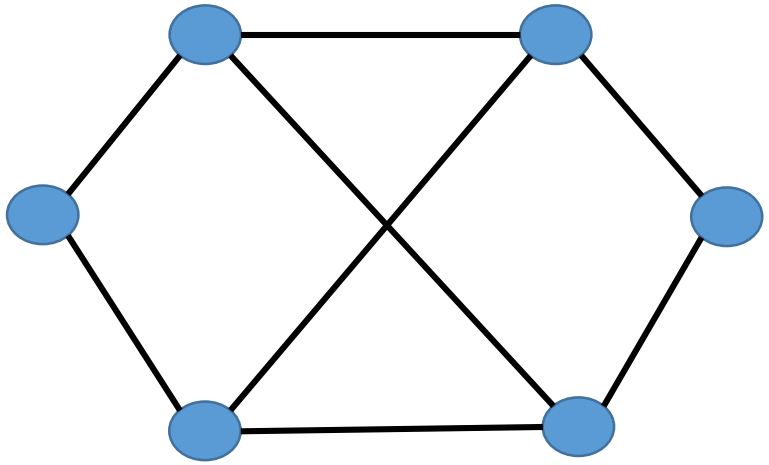
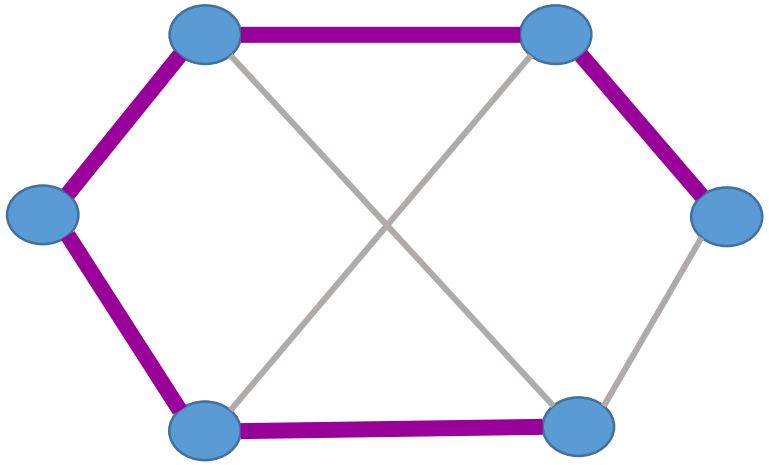
Network design

Goal: find a **low-cost subgraph** satisfying **nice** properties



Minimum spanning tree (MST)

Goal: find a minimum weight **connected** subgraph



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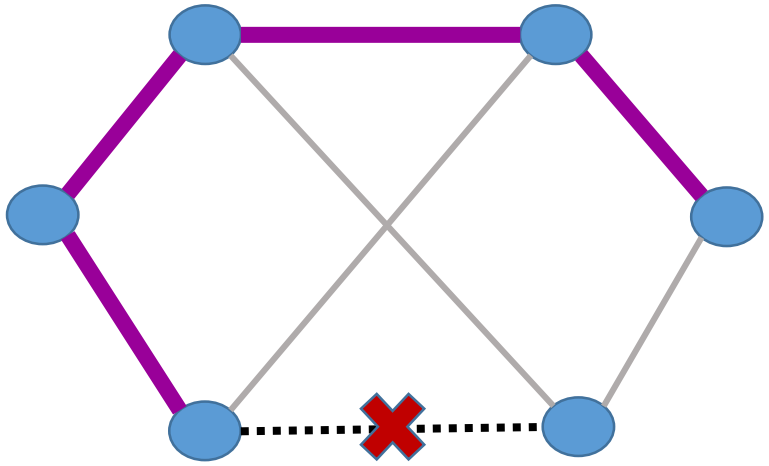
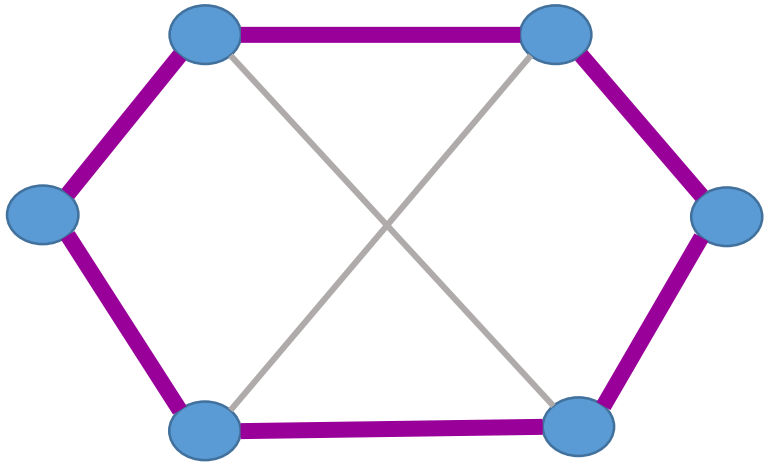


Cannot survive link failures!

Minimum 2-edge-connected spanning subgraph

Goal: find a minimum weight **2-edge-connected** subgraph



resistant to any *single* edge failure

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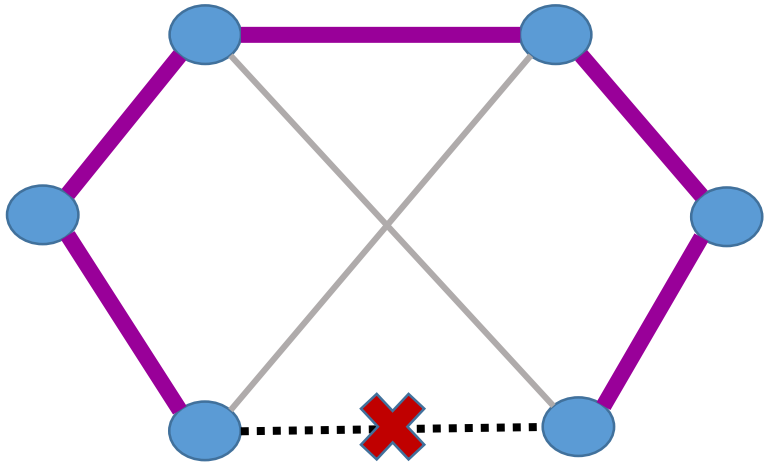

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Minimum **2-edge-connected** spanning subgraph

Our goal: find the minimum weight **2-edge-connected** spanning subgraph (*2-ECSS*)

- Central problem in **network design**
- Well-studied in the **sequential** setting: 2-approximations [Khuller and Vishkin 1994, Jain 2001]

Minimum 2-edge-connected spanning subgraph

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What about a **distributed** algorithm?

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What about a **distributed** algorithm?

CONGEST model:

n vertices, **synchronous** rounds, $\Theta(\log n)$-bit messages

Previous work & our results

Previous work

Round complexity	Approximation	Notes	Reference
$O(n)$	3	deterministic	Censor-Hillel and Dory, 17
$\tilde{\Omega}(D+\sqrt{n})$	Any polynomial α	randomized	Censor-Hillel and Dory, 17
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D = diameter

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Can we get the best of both worlds?

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Our Result: near-optimal algorithm

$\tilde{O}(D+\sqrt{n})$	$9+\epsilon$	deterministic	

*We recently improved the approximation to $5+\epsilon$

Our First Result: near-optimal algorithm

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Our Second Result: beyond worst-case graphs

Round complexity	Approximation	Graph Family
$\tilde{O}(D)$	$O(\log n)$	planar, bounded genus, bounded path-width, bounded tree-width
$\tilde{O}(D^2)$	$O(\log n)$	excluded minor
$2^{O(\sqrt{\log n})}$	$O(\log n)$	Erdos-Renyi random graphs

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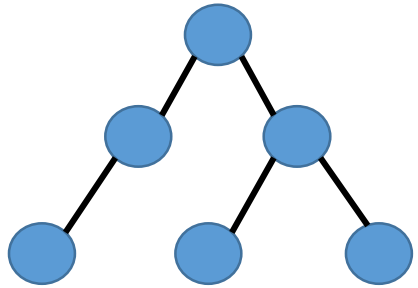
The Algorithm

The Algorithm

- We compute a minimum spanning tree T
- We **augment** its connectivity to 2



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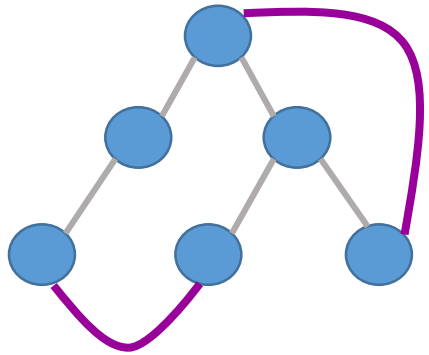
The tree augmentation problem (TAP):
Find a **minimum cost set of edges A** such that
$T \cup A$ is 2-edge-connected



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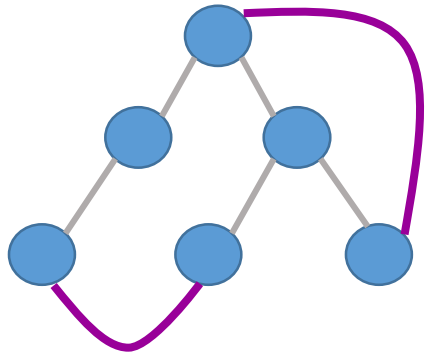

An α-approximation for **TAP** → An $(\alpha + 1)$-approximation for **2-ECSS**

TAP is a Set cover problem

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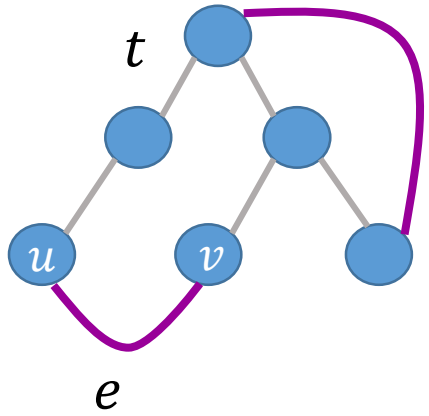
- A non-tree edge e **covers** a tree edge t if $(T \setminus t) \cup e$ is connected



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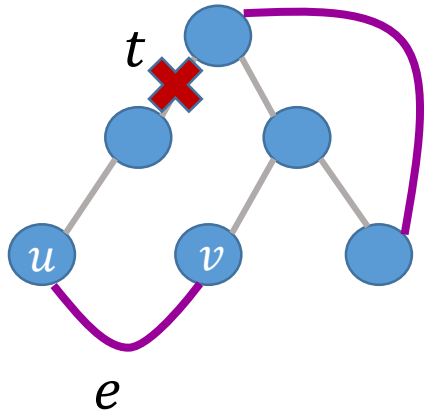

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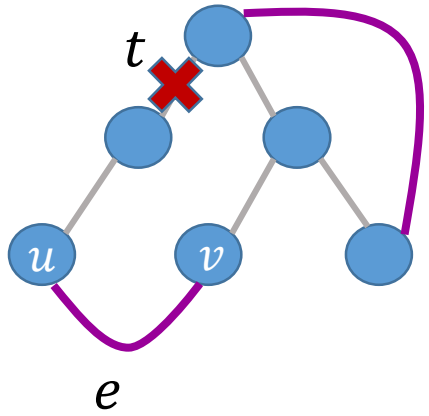

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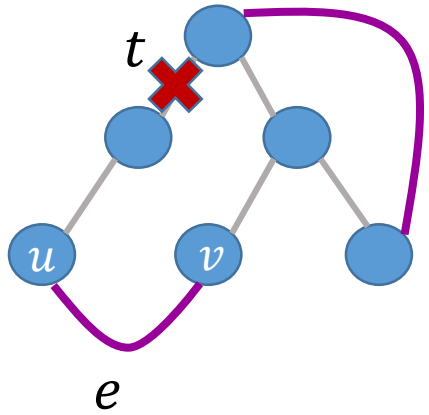
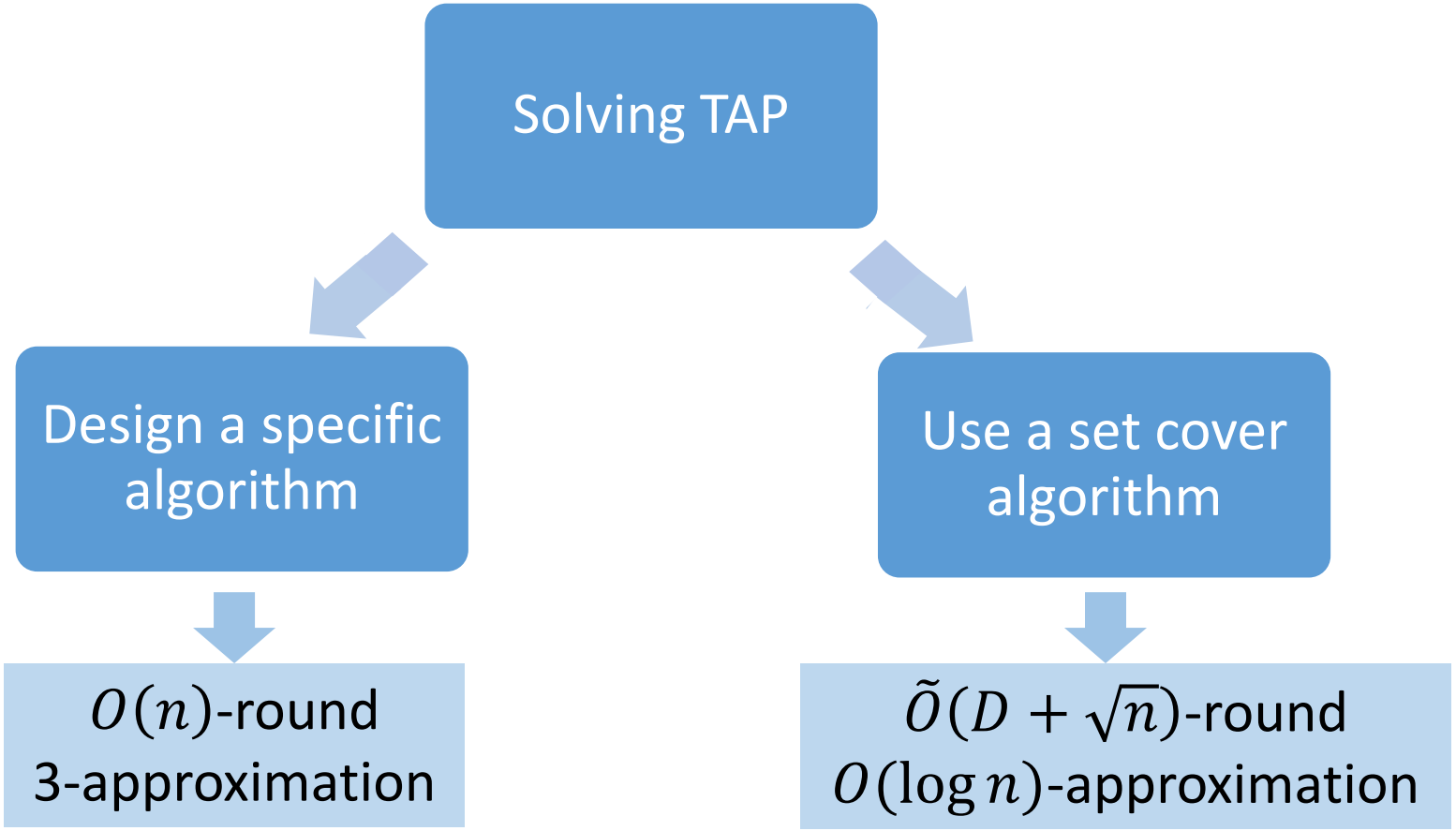
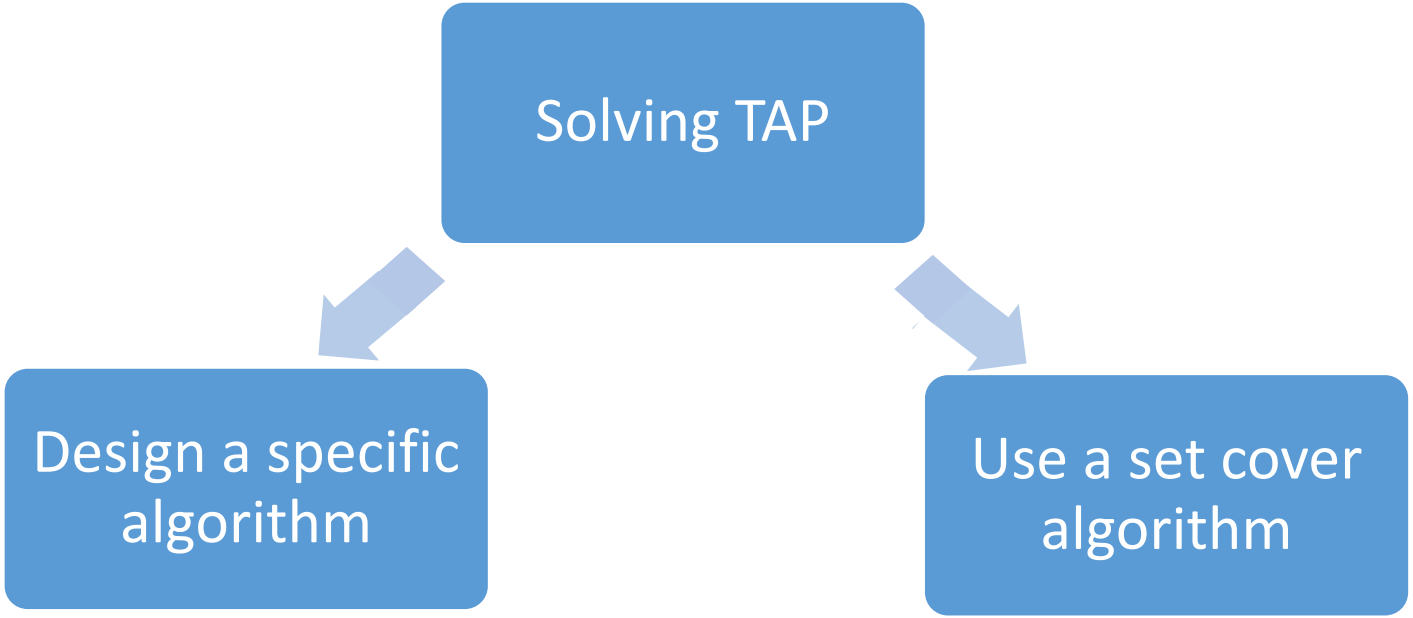
- Special case of set cover: cover a **universe** with minimum cost collection of **sets**




Solving TAP
Design a specific algorithm
Use a set cover algorithm
$O(n)$-round
3-approximation
$\tilde{O}(D+\sqrt{n})$-round
$O(\log n)$-approximation



Our approach:

Use a **set cover** algorithm that exploits the **specific structure** of TAP

$\tilde{O}(D + \sqrt{n})$-round $(9 + \epsilon)$-approximation

Exploiting the set cover structure

- Set cover with **small neighbourhood covers (SNC)**
- The SNC algorithm

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Set cover with **small neighbourhood covers** (SNC)

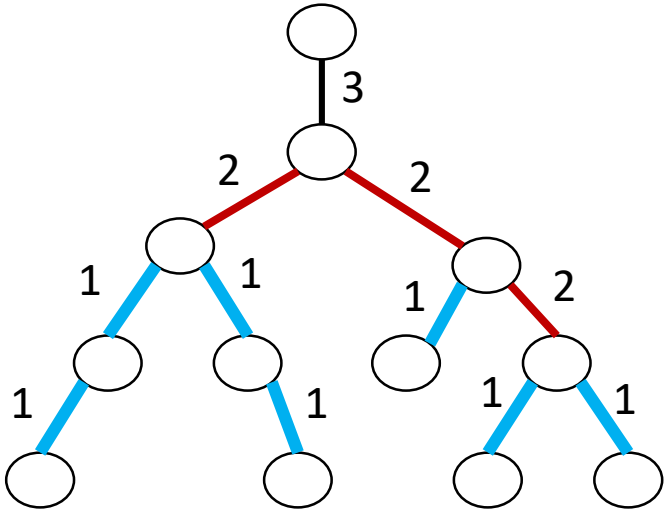
- The **SNC property** was introduced in

[Agarwal, Chakaravarthy, Choudhury, Roy, Sabharwal, 2018]

- Sequential and parallel algorithms that exploit this property
- Examples: vertex cover, interval cover, tree cover, …

Simplifying the problem

- It's "enough" to look at **paths**.
- We decompose the graph into $O(\log n)$ **layers** of disjoint paths.



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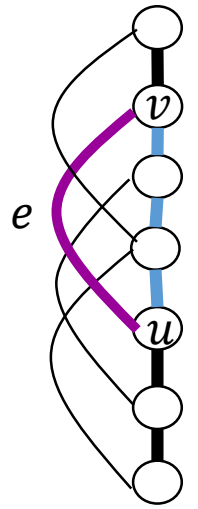

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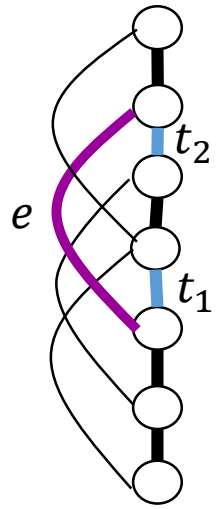

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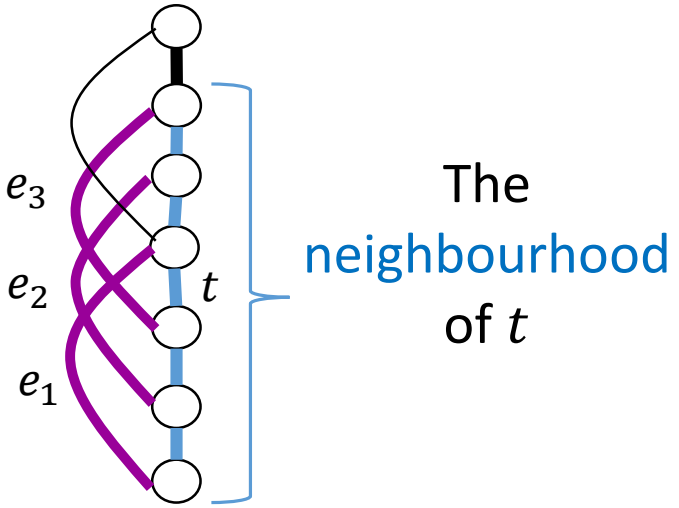

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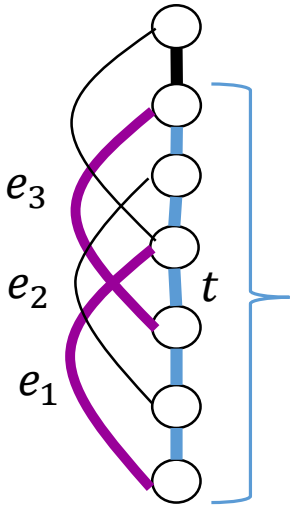

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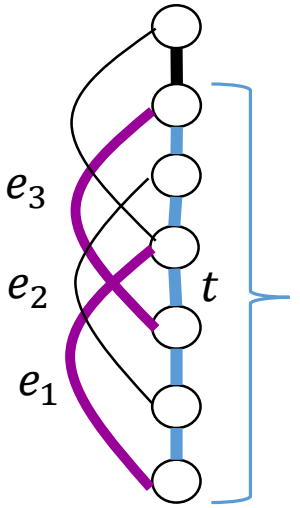
- These edges are called the **petals** of t.



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The SNC algorithm [Agarwal et al., 2018]

Forward phase:

Choose a set of "good" edges $\boldsymbol{A}$ that **covers all tree edges**

Reverse-delete phase:

Choose a cover $\boldsymbol{B} \subseteq A$ such that all tree edges are **covered at most 4 times** by B

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Gives a $(4+\epsilon)$-**approximation** for TAP in a path

Translates to a $(4+\epsilon)2+1 = (9+\epsilon)$-**approximation** for 2-ECSS

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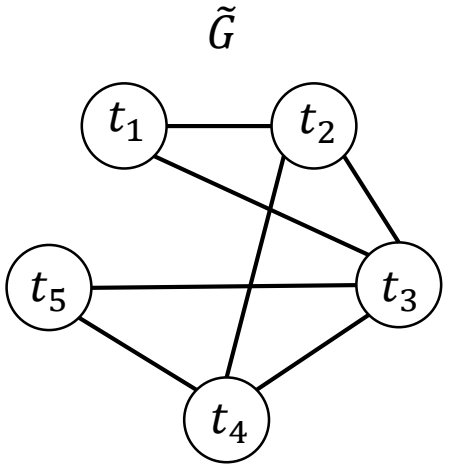

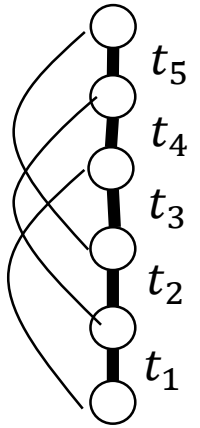

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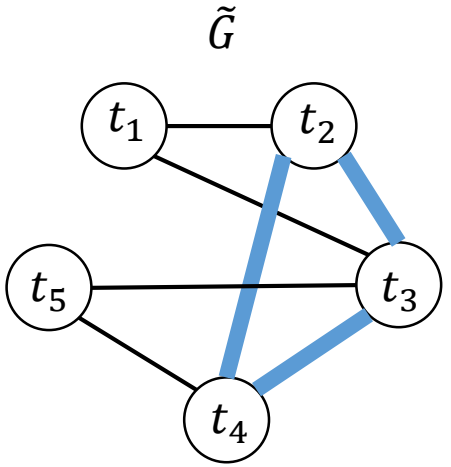

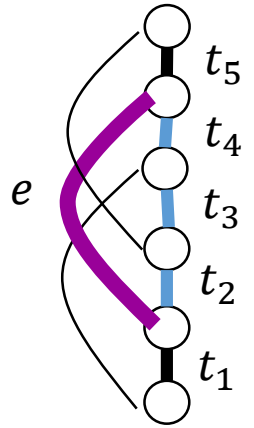

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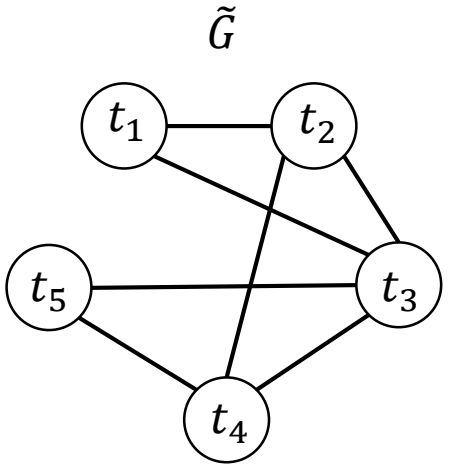

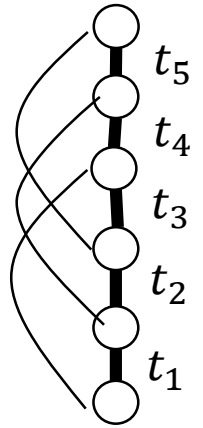

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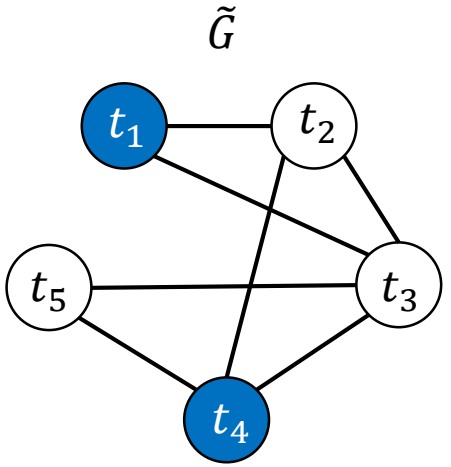

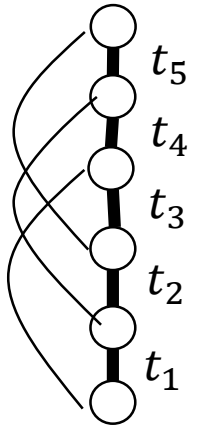

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The algorithm:

- Find a **maximal independent set** M in $\tilde{G}$
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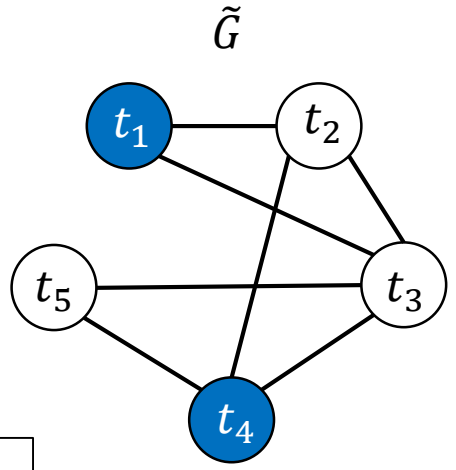

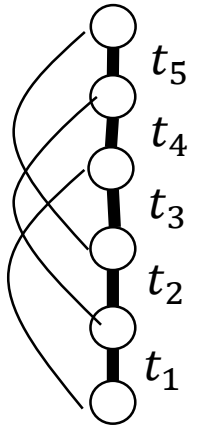

The **petals** of a tree edge t =
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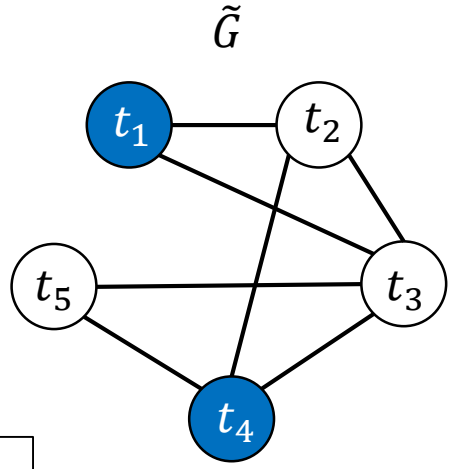

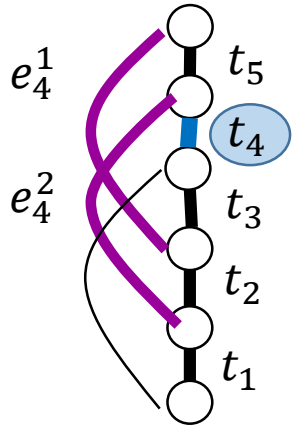

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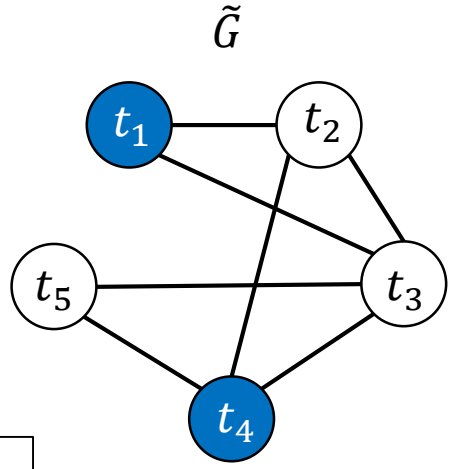

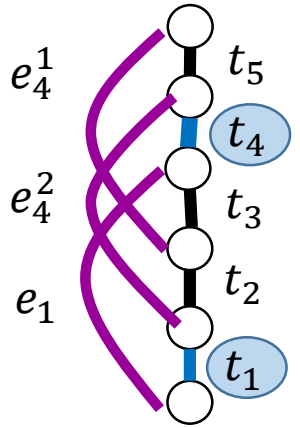

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The reverse-delete phase: correctness

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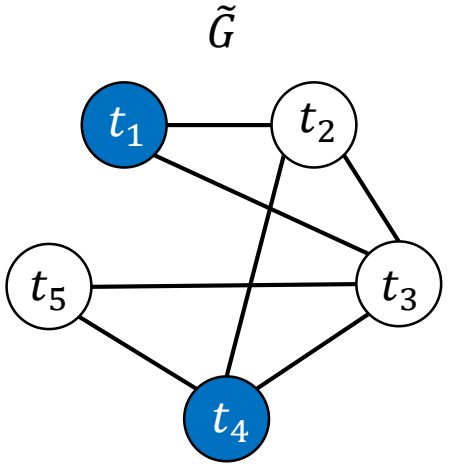
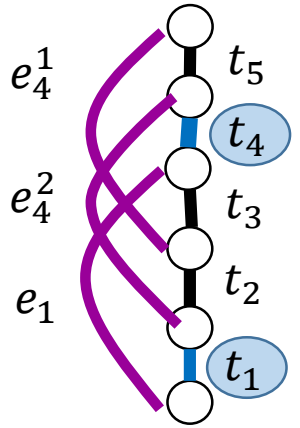
- For each $t \in M$, add its 2 **petals** to B





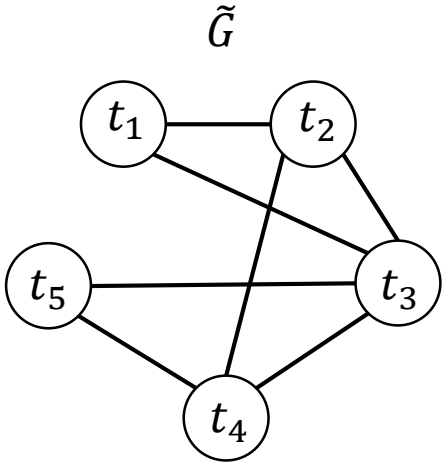
B is a cover:

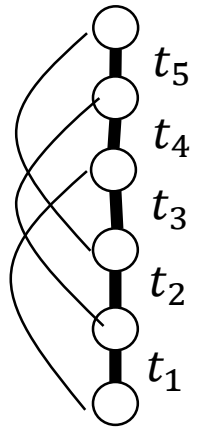
- Each tree edge t has a neighbor t' in M
- The petals of t' cover t

The reverse-delete phase: correctness

Choose a cover $\boldsymbol{B} \subseteq A$ such that all tree edges are **covered at most 4 times** by B

Claim: each tree edge t has at most 2 neighbors in M



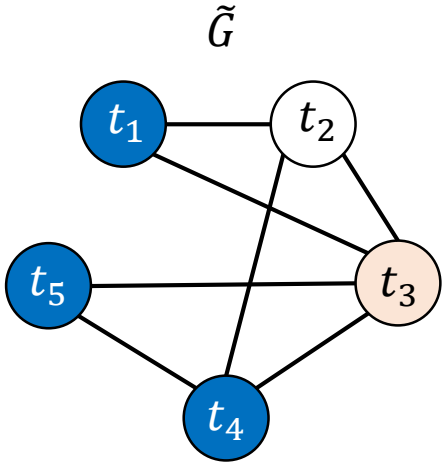


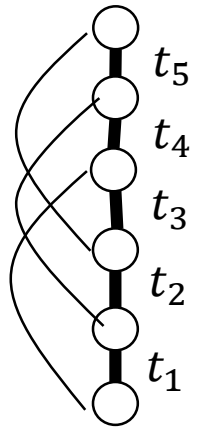
The reverse-delete phase: correctness

Choose a cover $B \subseteq A$ such that all tree edges are **covered at most 4 times** by B

Claim: each tree edge t has at most 2 neighbors in M

- Assume that t_3 has **3 neighbors** in M



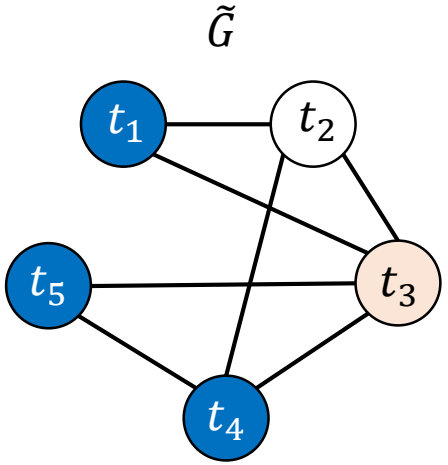


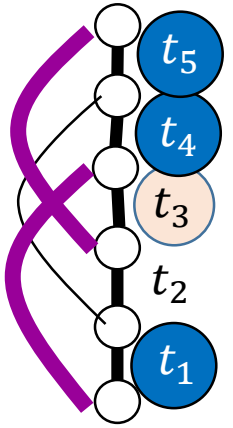
The reverse-delete phase: correctness

Choose a cover $B \subseteq A$ such that all tree edges are **covered at most 4 times** by B

Claim: each tree edge t has at most 2 neighbors in M

- Assume that t_3 has **3 neighbors** in M
- The **2 petals** of t_3 cover all its neighbors





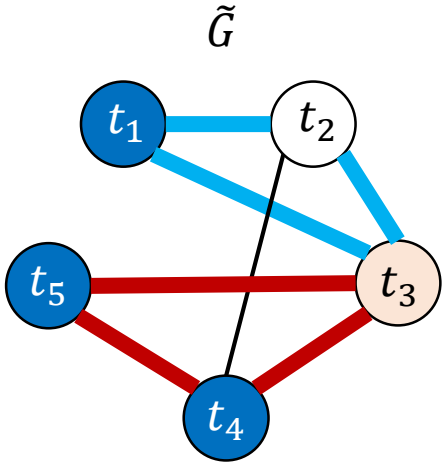
The reverse-delete phase: correctness

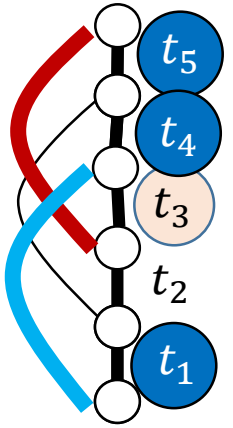
Choose a cover $\boldsymbol{B} \subseteq A$ such that all tree edges are **covered at most 4 times** by B

Claim: each tree edge t has at most 2 neighbors in M

- Assume that t_3 has **3 neighbors** in M
- The **2 petals** of t_3 cover all its neighbors

- At least **2 edges in M** are covered by the same petal → **neighbors**
- Contradiction to independence of M

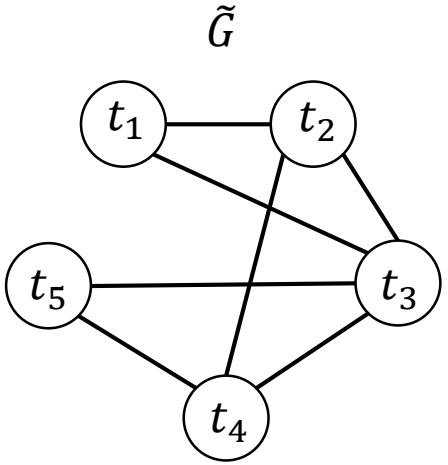


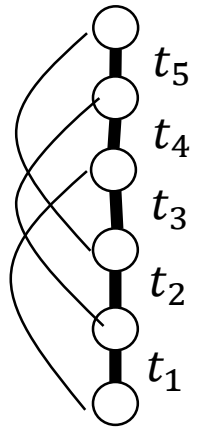


The reverse-delete phase: correctness

Choose a cover $\boldsymbol{B} \subseteq A$ such that all tree edges are **covered at most 4 times** by B

Claim: each tree edge t has at most 2 neighbors in M



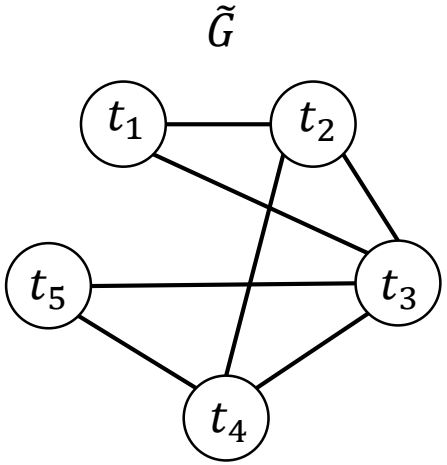


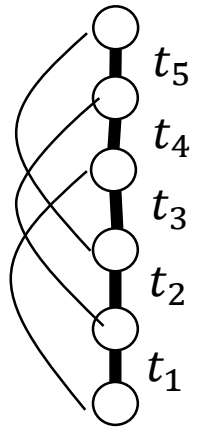
The reverse-delete phase: correctness

Choose a cover $B \subseteq A$ such that all tree edges are **covered at most 4 times** by B

Claim: each tree edge t has at most 2 neighbors in M

- Each of t's neighbors in M adds its **2 petals** to B
- t is covered at most **4 times**





The SNC algorithm [Agarwal et al., 2018]

Forward phase:
Choose a set of "good" edges A that **covers all tree edges**

Reverse-delete phase:
Choose a cover $B \subseteq A$ such that all tree edges are **covered at most 4 times** by B

Gives a $(4+\epsilon)$-**approximation** for TAP in a path

Translates to a $(4+\epsilon)2+1 = (9+\epsilon)$-**approximation** for 2-ECSS

Distributed implementation

Toolkit:

- LCA labels
- Decomposition
- Layering
- …

Main challenges:

- The **set cover** graph is not given
- Computing an MIS in the **neighborhood graph** $\tilde{G}$
- …

Our First Result: near-optimal algorithm

Round complexity	Approximation	Notes	
$\tilde{O}(D+\sqrt{n})$	$9+\epsilon$	deterministic	

Our Second Result: beyond worst-case graphs

Round complexity	Approximation	Graph Family
$\tilde{O}(D)$	$O(\log n)$	planar, bounded genus, bounded path-width, bounded tree-width
$\tilde{O}(D^2)$	$O(\log n)$	excluded minor
$2^{O(\sqrt{\log n})}$	$O(\log n)$	Erdos-Renyi random graphs

Open questions

- Minimum **k-edge-connected** subgraph for $k > 2$
- Exploiting the **specific structure** of set cover problems