Fast Approximate Shortest Paths in the Congested Clique

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Joint work with: Keren Censor-Hillel (Technion), Janne Korhonen (IST Austria), Dean Leitersdorf (Technion)

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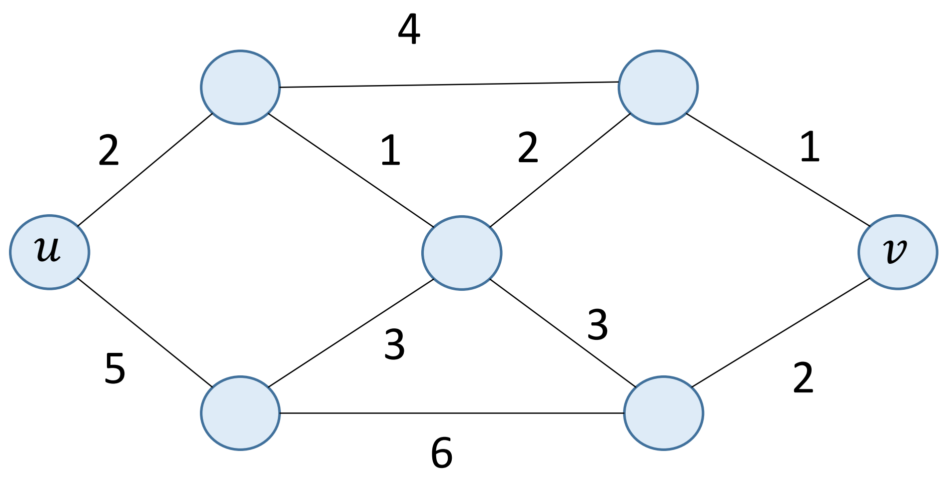
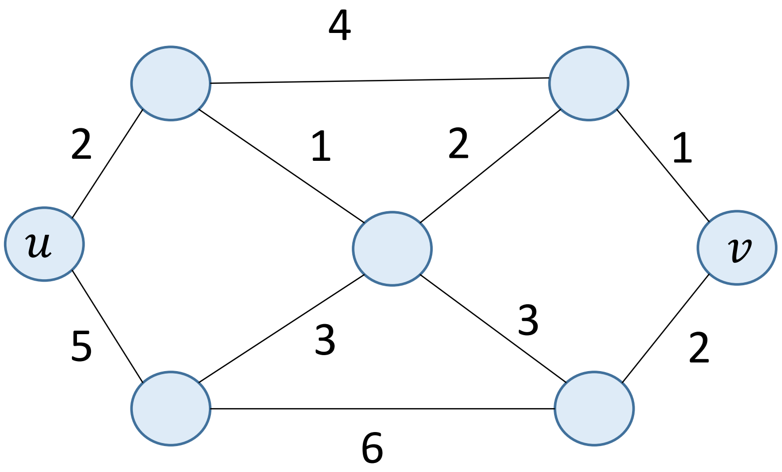
Distance Computation

- All-pairs shortest paths (APSP)
- Single-source shortest paths (SSSP)
- Multi-source shortest paths (MSSP)



The Congested Clique model

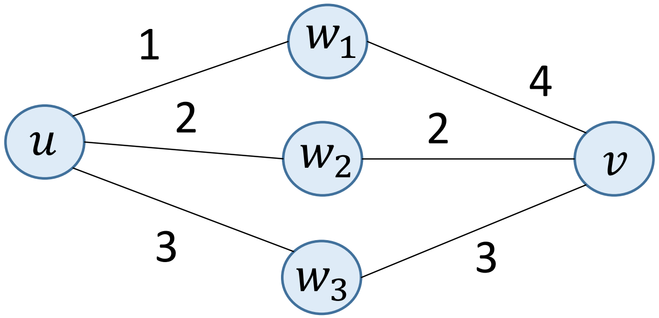
- n vertices
- Synchronous rounds
- $\Theta(\log n)$-bit messages to *all* vertices
- Input and output are *local*



Computing Distances using Matrix Multiplication

- A – weighted adjacency matrix
- Distance product:

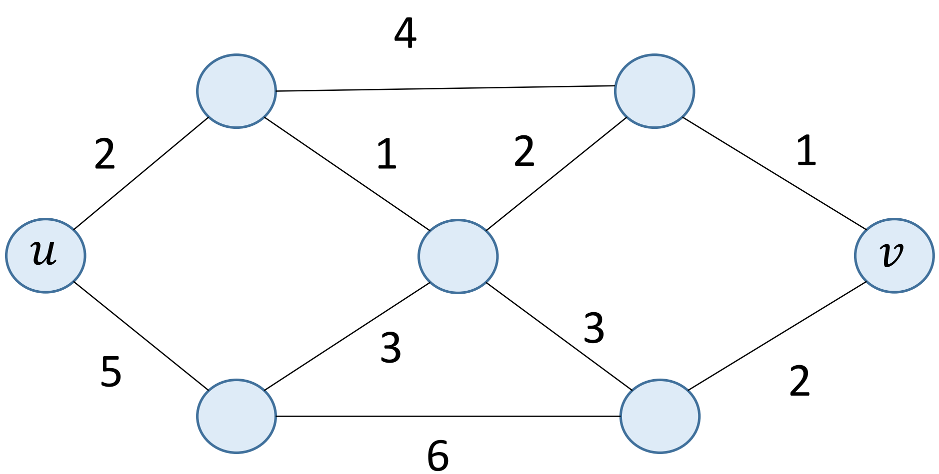
$$A^2[u,v] = \min_{w} A(u,w) + A(w,v)$$



- This is the minimum weight path between u and v of at most 2 edges

Computing Distances using Matrix Multiplication

- Similarly, $A^i[u, v]$ = minimum weight path between u and v of at most i edges (hops).
- Our goal: compute A^n



Computing Distances using Matrix Multiplication

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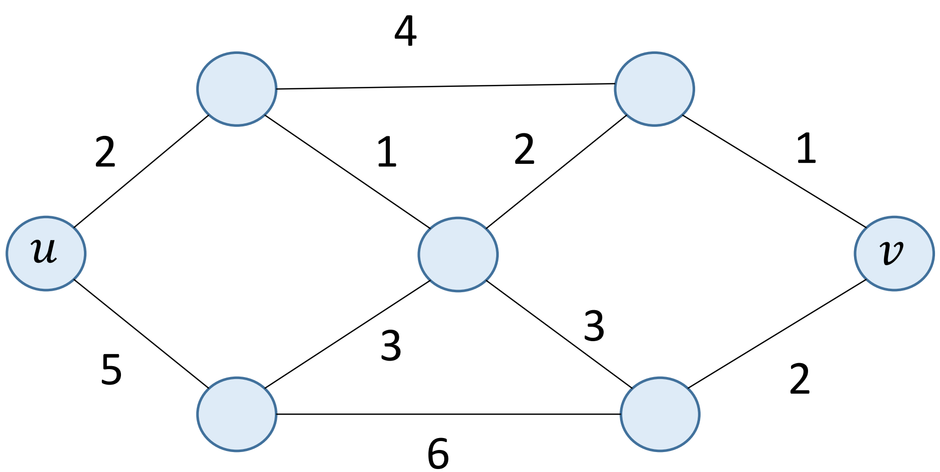
$$A[u,v] = \infty$$
$$A^2[u,v] = \infty$$
$$A^3[u,v] = 7$$
$$A^4[u,v] = 6$$
$$\dots$$
$$A^n[u,v] = 6$$



Computing Distances using Matrix Multiplication

- Our goal: compute A^n
- Requires $O(\log n)$ matrix multiplications:

$$A \rightarrow A^2 \rightarrow A^4 \rightarrow \ldots \rightarrow A^n$$

- How fast can we multiply matrices?

Computing Distances using Matrix Multiplication

$O\left(n^{1-2/\omega}\right) = O(n^{0.158})$	Ring	[Censor-Hillel, Kaski, Korhonen, Lenzen, Paz, Suomela '15]
$O\left(n^{1/3}\right)$	Semiring	[Censor-Hillel, Kaski, Korhonen, Lenzen, Paz, Suomela '15]
	Rectangular, Multiple instances, more	[Le Gall '16]

Computing Distances using Matrix Multiplication

$O(n^{0.158})$	• Exact **unweighted undirected** APSP • $\left(1 + o(1)\right)$-approximation for **weighted directed** APSP	[Censor-Hillel, Kaski, Korhonen, Lenzen, Paz, Suomela '15]
$\tilde{O}\left(n^{1/3}\right)$	Exact **weighted directed** APSP	[Censor-Hillel, Kaski, Korhonen, Lenzen, Paz, Suomela '15]
$O(n^{0.2096})$	Exact APSP in **directed** graphs with **constant** weights	[Le Gall '16]

Computing Distances using Matrix Multiplication

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All complexities are polynomial!

What about approximations?

- We can compute a **spanner**: a sparse subgraph that approximates the distances.

$\tilde{O}(n^{1/k})$	$(2k-1)$-approximation for **weighted undirected** APSP

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Still polynomial for any constant k!

Computing Distances in the Congested Clique

Can we get constant approximation for APSP in sub-polynomial time?

Computing Distances in the Congested Clique

Can we get constant approximation for APSP
in sub-polynomial time?

- <u>For SSSP:</u>

$O(\epsilon^{-3} \text{polylog } n)$-round $(1+\epsilon)$-approximation
[Becker, Karrenbauer, Krinninger, Lenzen '17]

Computing Distances in the Congested Clique

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- <u>For SSSP:</u>

$O(\epsilon^{-3}\text{polylog}\, n)$-round $(1+\epsilon)$-approximation
[Becker, Karrenbauer, Krinninger, Lenzen '17]

Only for a single source!

Our Results: APSP

$O(\log^2 n/\epsilon)$	• $(2+\epsilon)$-approximation for **unweighted undirected** APSP • $(3+\epsilon)$-approximation for **weighted undirected** APSP

First polylog constant-factor approximation!

$(2-\epsilon)$-APSP implies MM	[Dor, Halperin, Zwick ‘00 Korhonen, Suomela ‘18]

Our Results: MSSP and more

$O(\log^2 n/\epsilon)$	$(1+\epsilon)$-approximation weighted undirected **MSSP** with $O(n^{1/2})$ sources
$O(\log^2 n/\epsilon)$	Near $(3/2)$-approximation for **diameter**
$\tilde{O}(n^{1/6})$	Exact weighted undirected **SSSP**

Previous results:

$\tilde{O}(n^{1/3})$	Exact weighted **SSSP**	[Censor-Hillel, Kaski, Korhonen, Lenzen, Paz, Suomela '15]
$O(\epsilon^{-3}\text{polylog}\, n)$	$(1+\epsilon)$-**SSSP**	[Becker, Karrenbauer, Krinninger, Lenzen '17]

Our Techniques

- We can multiply *sparse* matrices faster:

$O\left(1+\frac{(\rho_S\rho_T)^{1/3}}{n^{1/3}}\right)$	Semiring, Sparse	[Censor-Hillel, Leitersdorf, Turner '18]

- ρ_A= density of A, the average number of non-zero entries on a row
- Example: $O(1)$ rounds for $O(n^{3/2})$ edges.

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- How can we use this?

  - Even if A is sparse, A^2 can be dense.

  - We want to compute distances in *general* graphs.

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- How can we use this?

  - Even if A is sparse, A^2 can be dense.

  - We want to compute distances in *general* graphs.

Many *building blocks* for distance computation are actually based on computations in *sparse* graphs

Building blocks for distance computation

- k-nearest: for each vertex, compute distances to k nearest vertices

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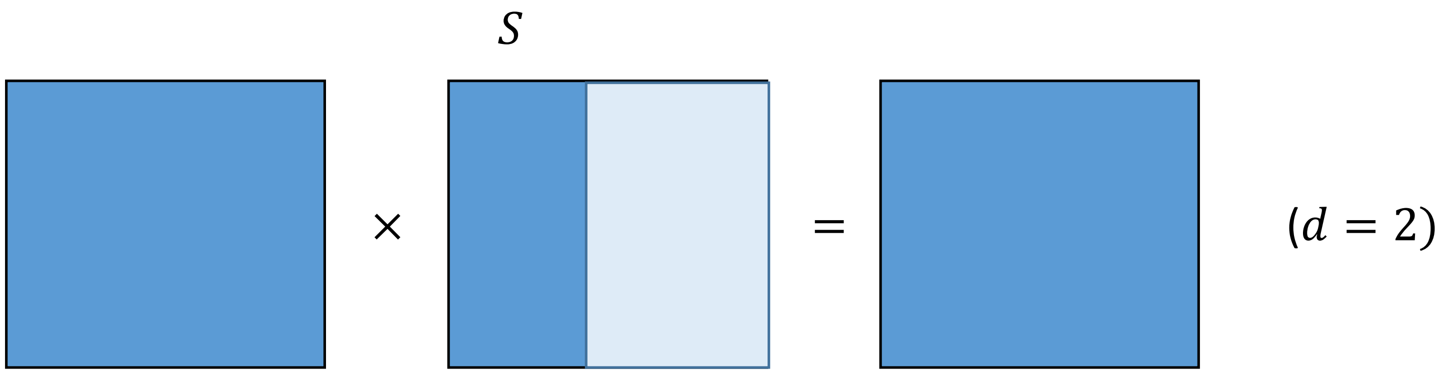
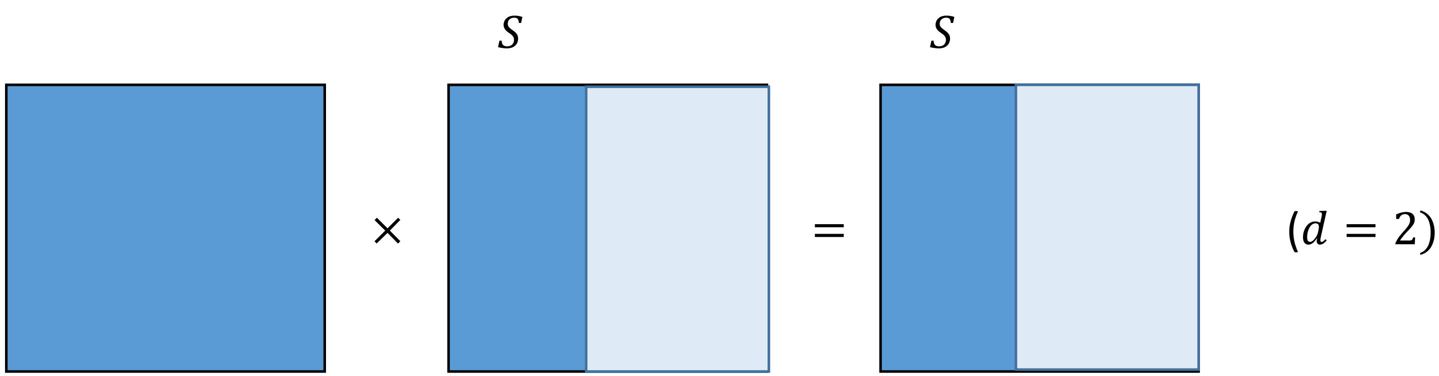


Multiplication of sparse matrix by dense matrix: previous MM algorithm is still polynomial

Building blocks for distance computation

- (S, d, k)-source detection: for each vertex, compute distances to k nearest sources in S, up to hop d



Output matrix is also sparse!

Building blocks for distance computation

- k-nearest: for each vertex, compute distances to k nearest vertices
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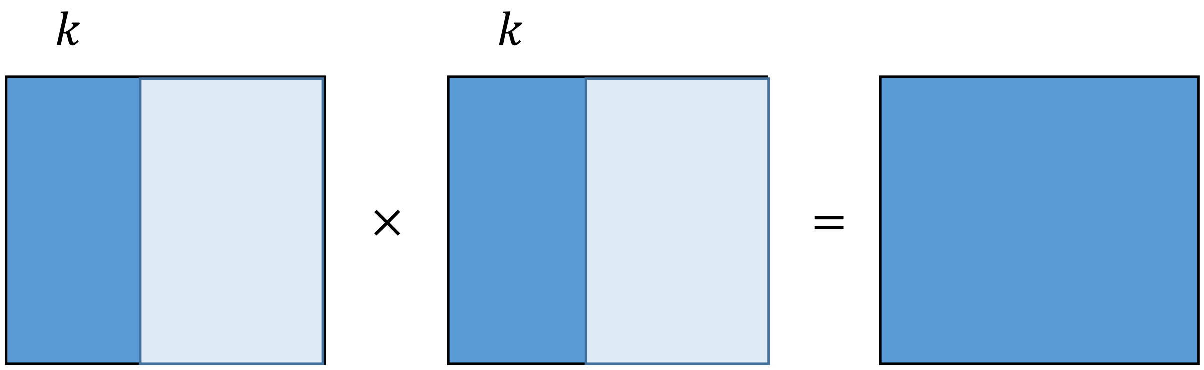
Building blocks for distance computation

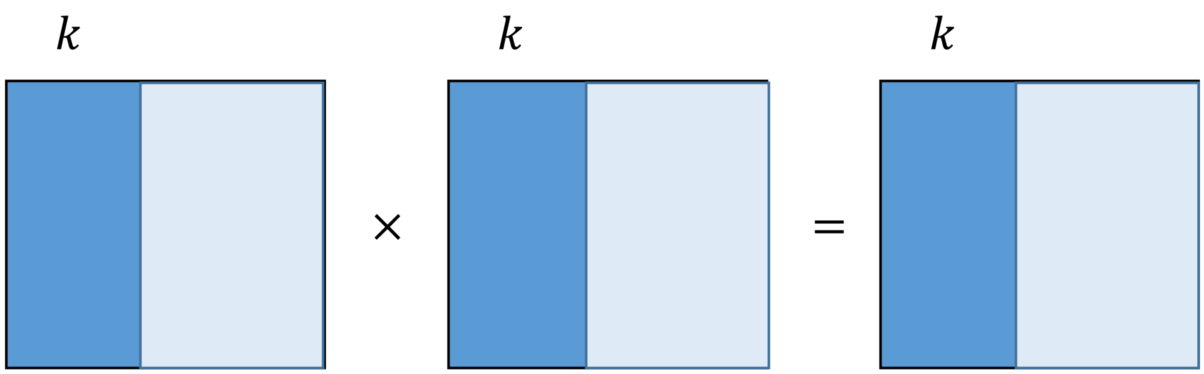
- *k-nearest*: for each vertex, compute distances to k nearest vertices



It's enough to look only at the k closest vertices to each vertex

Building blocks for distance computation

- k-nearest: for each vertex, compute distances to k nearest vertices



It's enough to look only at the k closest vertices to each vertex: also in the *output*

We don't know the identity of the k closest vertices before the computation

New Matrix Multiplication algorithm

- Previous algorithm:

$O\left(1+\frac{(\rho_S\rho_T)^{1/3}}{n^{1/3}}\right)$	Semiring, Sparse	[Censor-Hillel, Leitersdorf, Turner '18]

- Our algorithm:

$O\left(1+\frac{(\rho_S\rho_T\rho_P)^{1/3}}{n^{2/3}}\right)$	Semiring, Sparse	[Censor-Hillel, Dory, Korhonen, Leitersdorf, '19]

$$S \times T = P$$

New Matrix Multiplication algorithm

- Our algorithm:

$O\left(1+\frac{(\rho_S\rho_T\rho_P)^{1/3}}{n^{2/3}}\right)$	Semiring, Sparse	[Censor-Hillel, Dory, Korhonen, Leitersdorf, '19]

- Depends also on the sparsity of the *output* matrix
- Even if we don't know the structure of the output matrix, we can *sparsify* the output matrix *on-the-fly*, keeping only ρ_P smallest entries for each row

Application: Distance Tools

k-nearest:

$O\left(\left(\frac{k}{n^{2/3}}+1\right)\log k\right)$ rounds ⇨ $O(\log n)$ for $k = n^{2/3}$

(S, d, k)-source detection:

$O\left(\left(\frac{m^{1/3}k^{2/3}}{n}+1\right)d\right)$ rounds ($m =$ number of edges)

Work for directed weighted graphs

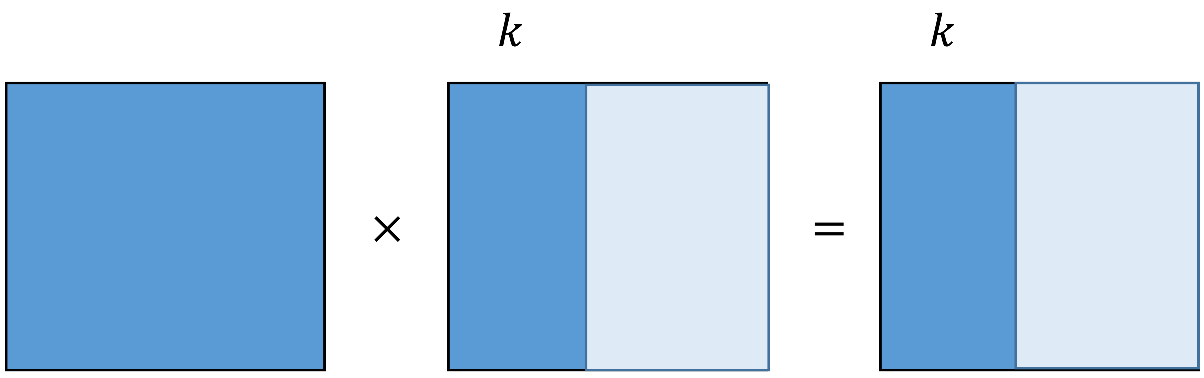
$O\left(1+\frac{(\rho_S\rho_T\rho_P)^{1/3}}{n^{2/3}}\right)$	Semiring, Sparse	[Censor-Hillel, Dory, Korhonen, Leitersdorf, '19]

Application: Distance Tools

(S, d, k)-source detection:

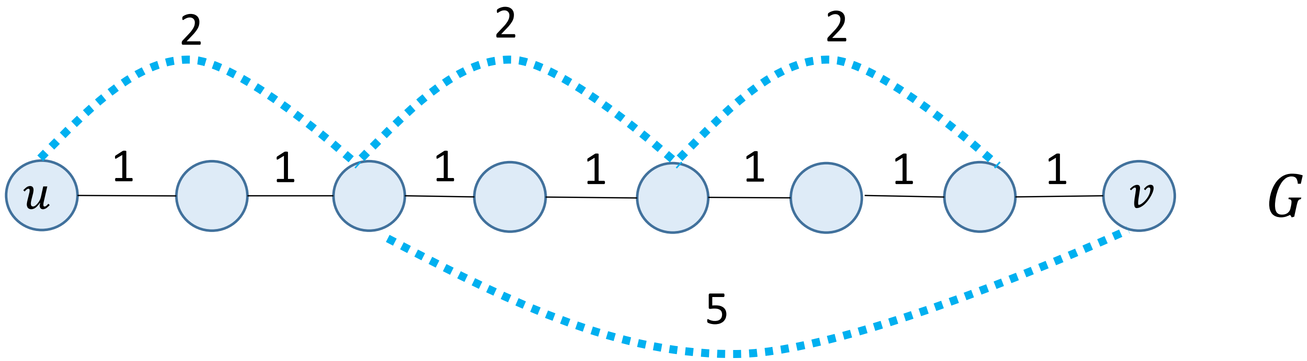
$O\left(\left(\frac{m^{1/3}k^{2/3}}{n} + 1\right) d\right)$ rounds (m = number of edges)

- To exploit the sparsity we need $\boldsymbol{d = n}$ multiplications - too expensive!



Solution: Hopsets

(β, ϵ)-hopset H:
A graph $H = (V, E')$, such that the β-hop distances in $G \cup H$ give $(1+\epsilon)$-approximation for the distances in G



Enough to look at β-hop distances!

Solution: Hopsets

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Our goal: to have small β and small running time t

What is known?

- We can get $\beta = t = O\left(\frac{\log\log n}{\epsilon}\right)^{\log\log n}$
  [Elkin, Neiman '17]

Solution: Hopsets

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Can we get a poly-logarithmic complexity?

Solution: Hopsets

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- We can get $\beta = t = O\left(\frac{\log\log n}{\epsilon}\right)^{\log\log n}$ [Elkin, Neiman '17]

Yes! we can get: $\beta = O\left(\frac{\log n}{\epsilon}\right), t = O\left(\frac{\log^2 n}{\epsilon}\right)$

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- We can get: $\beta = O\left(\frac{\log n}{\epsilon}\right)$, $t = O\left(\frac{\log^2 n}{\epsilon}\right)$

Idea: using our *distance tools* we can implement efficiently the hopset construction of [Elkin, Neiman '17] [Huang, Pettie '19] [Thorup, Zwick '06]

Applications: MSSP

(β, ϵ)-hopset H:
A graph $H = (V, E')$, such that the β-hop distances in $G \cup H$ give $(1+\epsilon)$-approximation for the distances in G

(S, d, k)-source detection: $O\left(\left(\frac{m^{1/3}k^{2/3}}{n} + 1\right) d\right)$ rounds

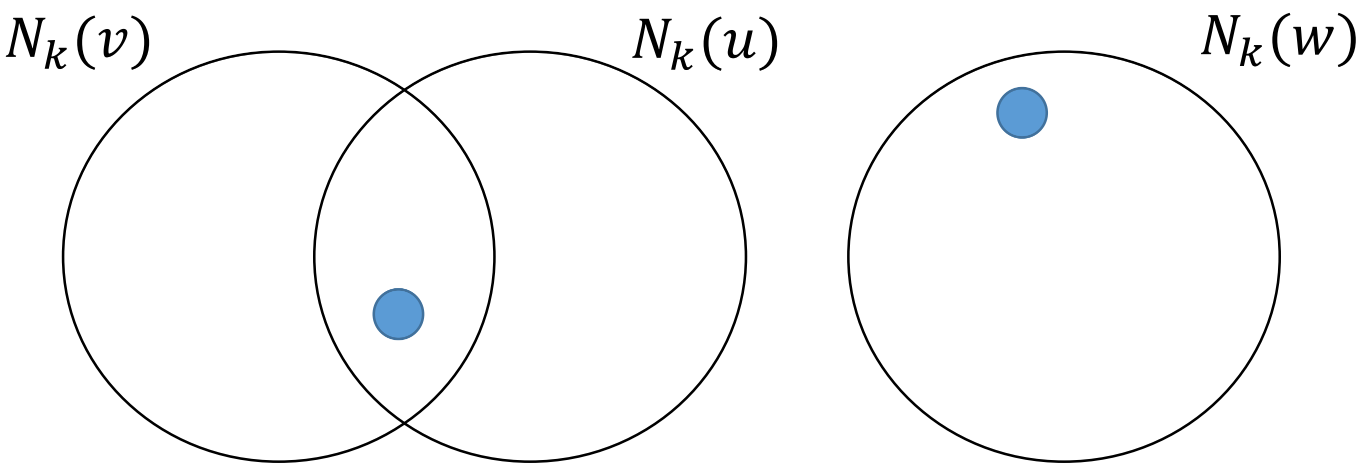
- We run our (S, d, k)- source detection in $G \cup H$ with $d = \beta$: $(1+\epsilon)$–approximation for the distances of all vertices from S

Complexity: $O\left(\left(\frac{|S|^{2/3}}{n^{1/3}} + \log n\right) \frac{\log n}{\epsilon}\right)$

➡ poly-logarithmic for $|S| = O(\sqrt{n})$

Applications: APSP

- We compute the $k = \tilde{O}(\sqrt{n})$ nearest vertices $N_k(v)$ for each v
- We compute a hitting set $\boldsymbol{A}$ of the sets $N_k(v)$ with $|A| = \tilde{O}(\sqrt{n})$



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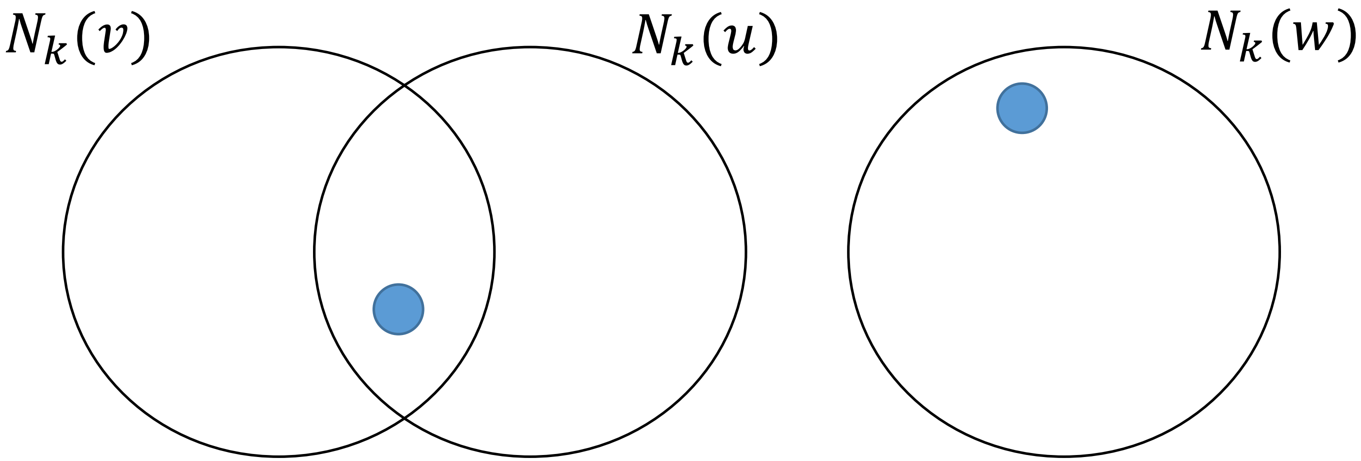
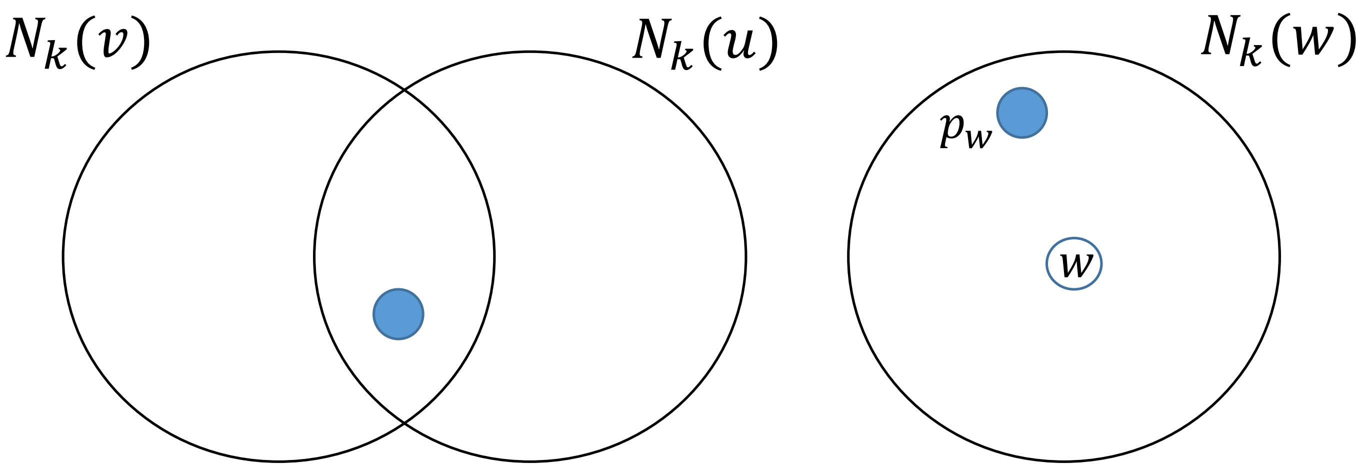
- We compute a hitting set $\boldsymbol{A}$ of the sets $N_k(v)$ with $|A| = \tilde{O}(\sqrt{n})$: $O((\log\log n)^3)$ rounds [Parter, Yogev '18]



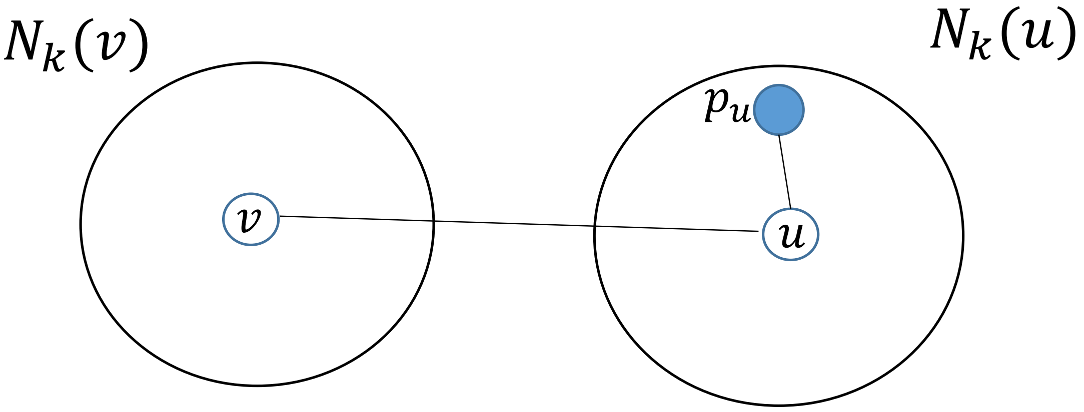
Applications: APSP

- For each v, let p_{v} = the closest vertex to v in $A \cap N_k(v)$
- We compute $(1+\epsilon)$-approximate distances $\delta(u,v)$ from all vertices to A



Applications: APSP

- If $v \in N_k(u)$, u knows $d(u,v)$.
- Otherwise, we estimate $\delta(u,v) = \delta(u,p_u) + \delta(p_u,v)$



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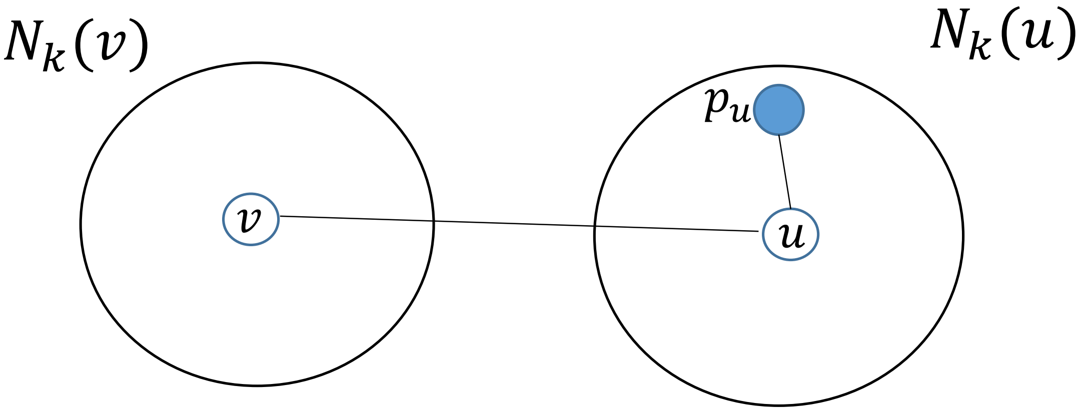

- $d(u,p_u) \le d(u,v)$

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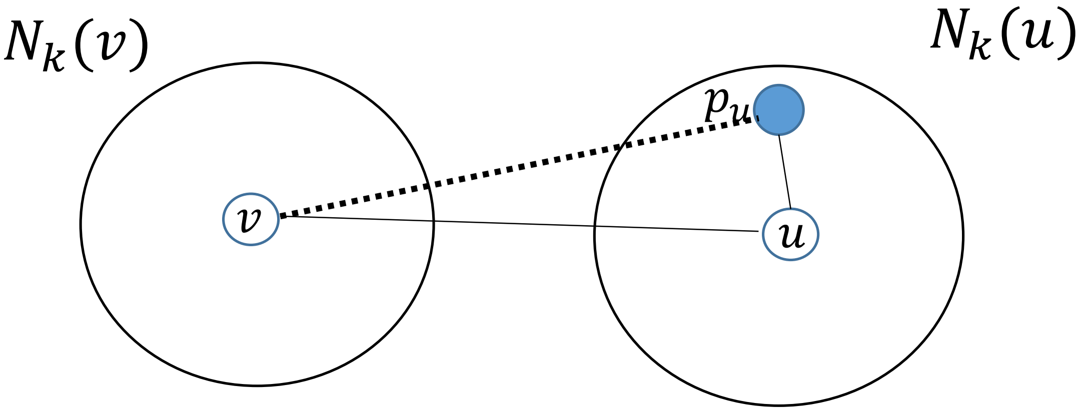


- $d(u,p_u) \le d(u,v)$
- $d(v,p_u) \le d(u,v) + d(u,p_u) \le 2d(u,v)$
- $\delta(u,p_u) + \delta(p_u,v)$ gives a $(\mathbf{3} + \boldsymbol{\epsilon})$-approximation

Applications: APSP

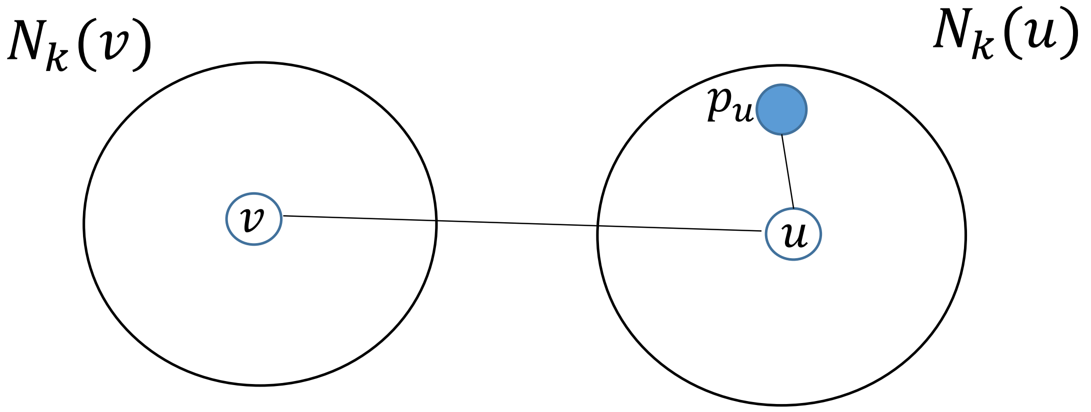
<u>Complexity:</u>

- Computing the k-nearest: $O(\log n)$ time
- Computing a hitting set: $O((\mathrm{loglog}n)^3)$ time
- Computing distances to A: $O\left(\frac{\log^2 n}{\epsilon}\right)$ time

$(3+\epsilon)$-approximation in $O\left(\frac{\log^2 n}{\epsilon}\right)$ rounds

Applications: APSP

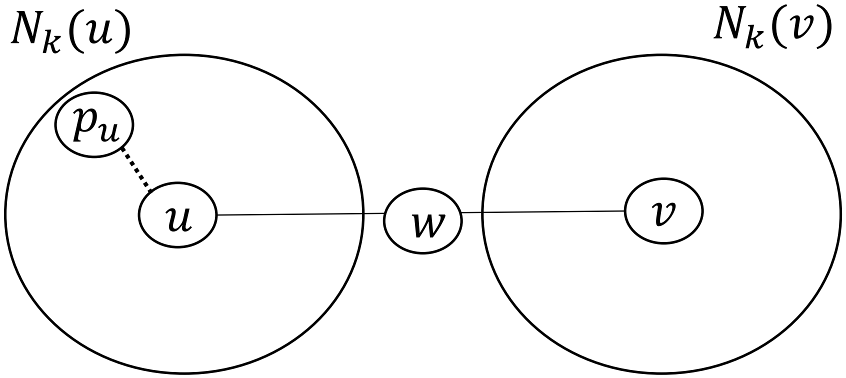
Can we improve the approximation?



If $d(u,p_u) \leq \frac{d(u,v)}{2}$ then the same analysis shows a $(2+\epsilon)$-approximation

Applications: APSP

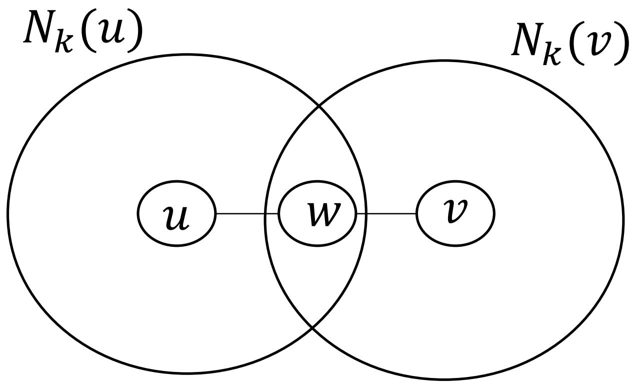
<u>Case 1:</u>



$d(u, p_u) \leq \frac{d(u,v)}{2}$ ⇒ $(2 + \epsilon)$-approximation

Applications: APSP

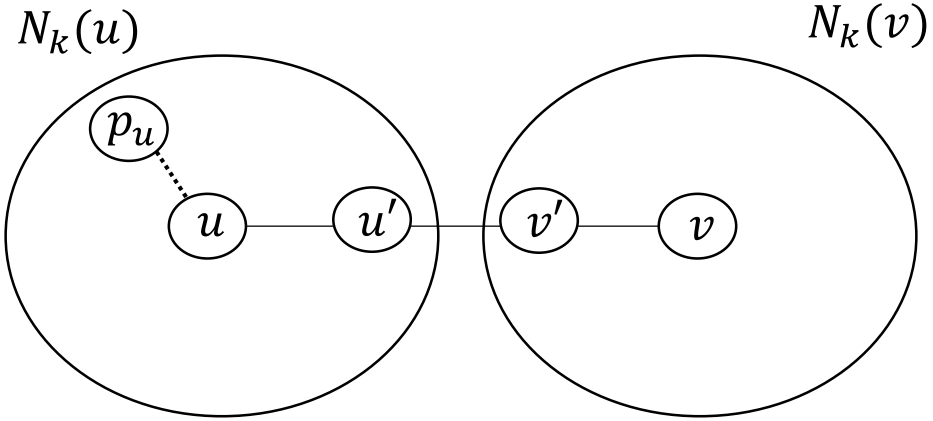
<u>Case 2:</u>



$w \in N_k(u) \cap N_k(v)$ ⇒ We can compute $d(u, v)$

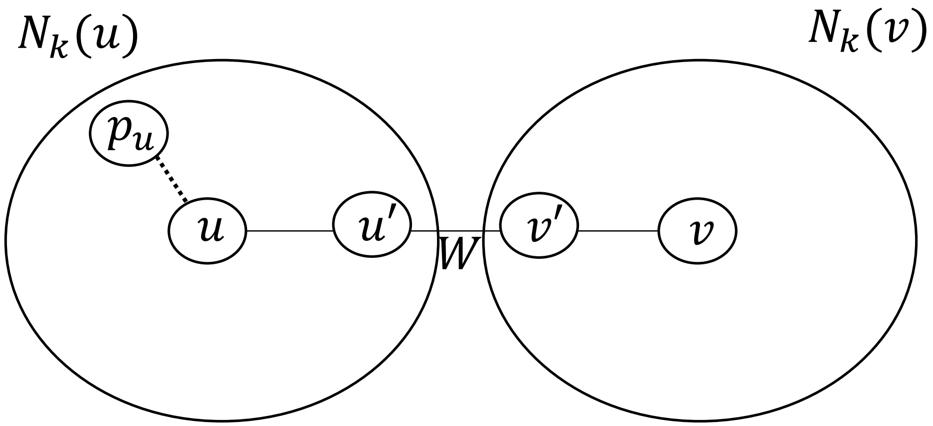
Applications: APSP

<u>Case 3:</u>



Applications: APSP

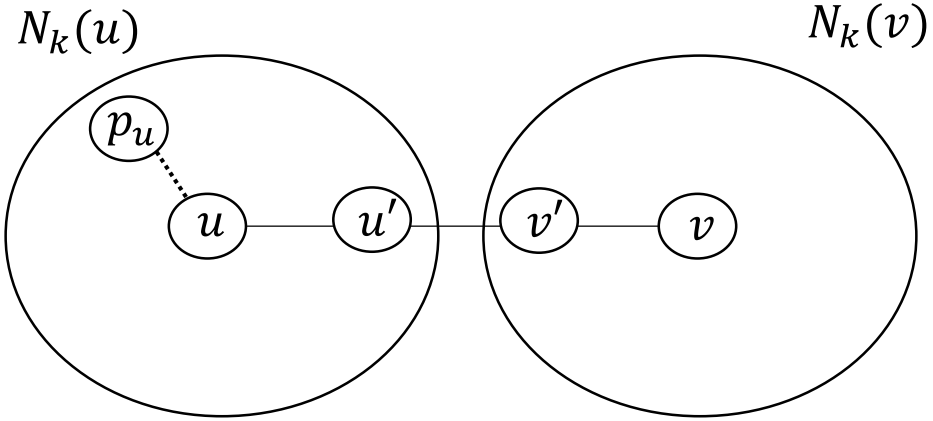
Case 3:



$d(u, p_u) \leq \frac{d(u,v)+\boldsymbol{W}}{2}$ ⇨ $\left((2+\epsilon), (1+\epsilon)\boldsymbol{W}\right)$-approximation

Applications: APSP

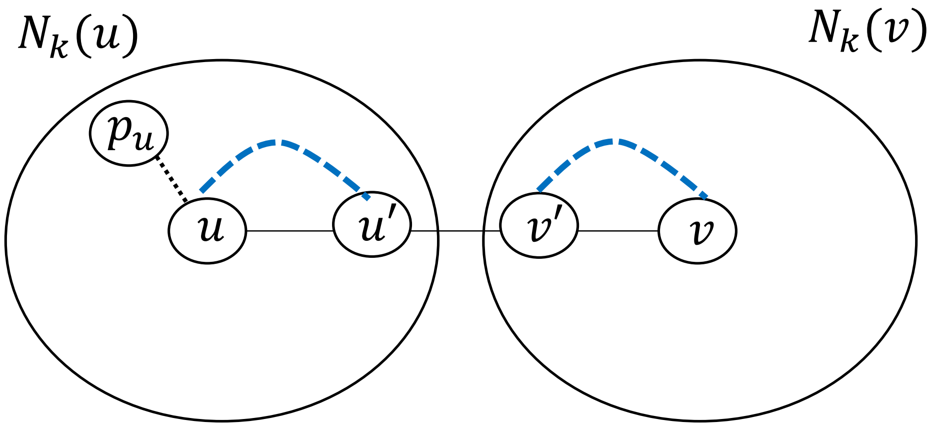
Case 3:



How do we get a $(2 + \epsilon)$-approximation?

Applications: APSP

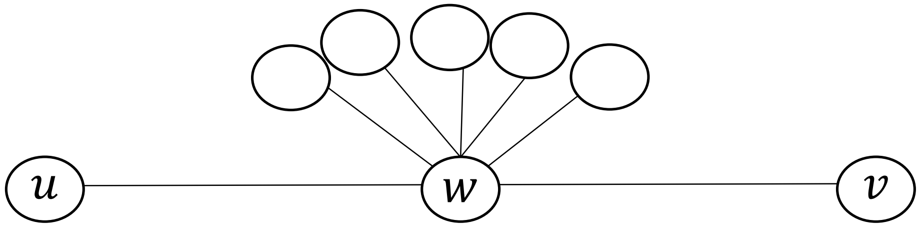
<u>Case 3:</u>



- We have a 3-hop shortest path between u and v
- However, the 3 relevant matrices are too dense

Applications: APSP

<u>Dense paths:</u>



$\deg(w) \geq \sqrt{n}$

Applications: APSP

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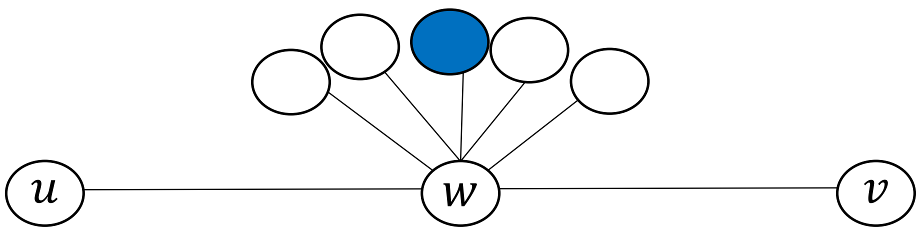
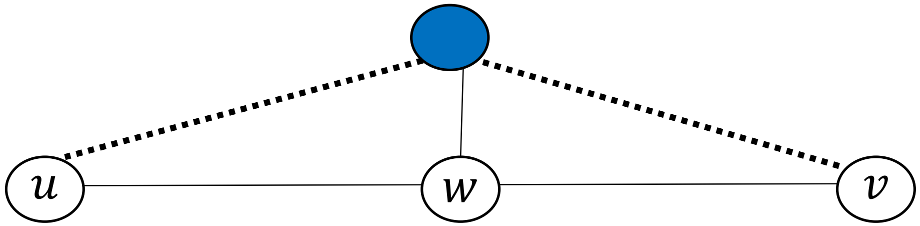


$\deg(w) \geq \sqrt{n}$

Compute a hitting set of size $\tilde{O}(\sqrt{n})$ for high-degree vertices

Applications: APSP

<u>Dense paths:</u>

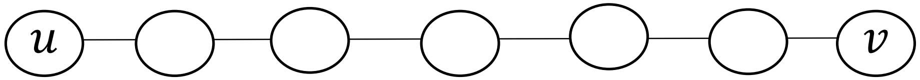


$\deg(w) \geq \sqrt{n}$

The distance through the hitting set gives a +2 approximation

Applications: APSP

Sparse paths:

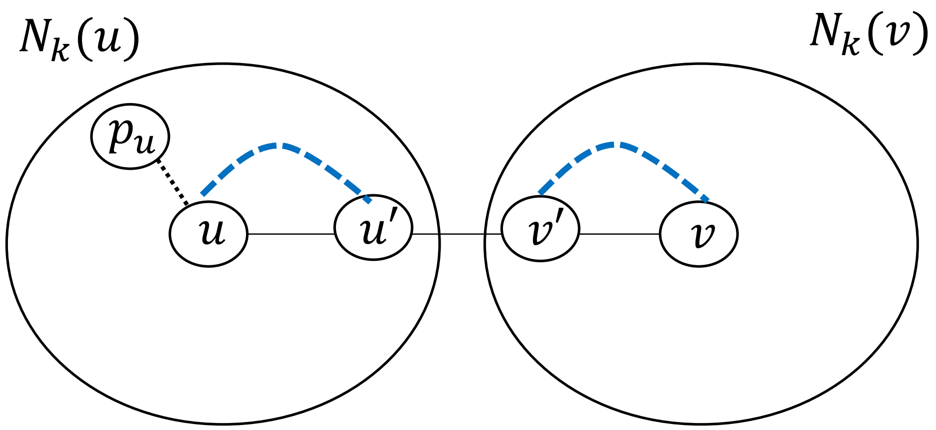


- All degrees are at most $\sqrt{n}$
- We can focus on a sparse graph with $O(n^{3/2})$ edges.

Applications: APSP

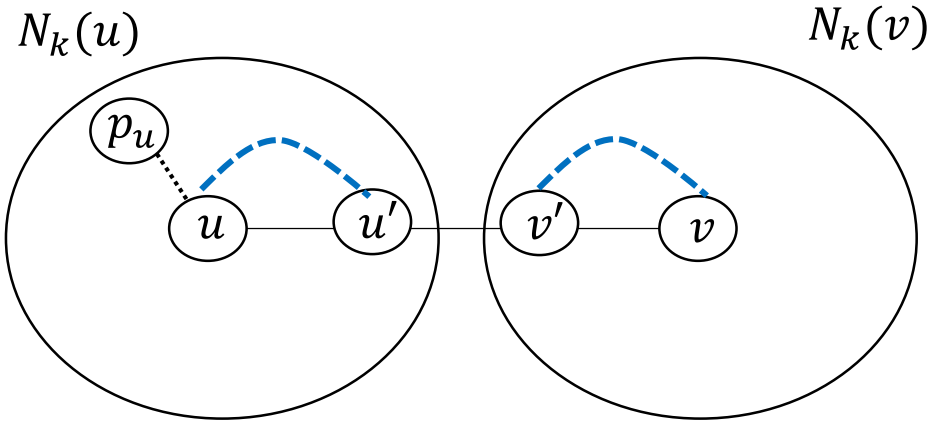
<u>Sparse paths:</u>

- We can focus on a sparse graph with $\boldsymbol{O(n^{3/2})}$ edges.
- We can compute MSSP from $\boldsymbol{\widetilde{O}(n^{3/4})}$ sources.
- Can take $k = \boldsymbol{\widetilde{O}(n^{1/4})}$.
- Now the 3 relevant matrices are sparse enough.



Applications: APSP

$(2+\epsilon)$-approximation for **unweighted** APSP in $O\left(\frac{\log^2 n}{\epsilon}\right)$ rounds



Conclusion

- We show a fast algorithm for matrix multiplication that depends on the *sparsity* and is *output*-sensitive.
- Allows to build efficient *distance tools.*
- Together with *hopsets*: polylog algorithms for MSSP, APSP.

Summary

$O(\log^2 n/\epsilon)$	• $(2+\epsilon)$-approximation for **unweighted undirected** APSP • $(3+\epsilon)$-approximation for **weighted undirected** APSP
$O(\log^2 n/\epsilon)$	$(1+\epsilon)$-approximation for **weighted** undirected **MSSP** with $O(n^{1/2})$ sources
$O(\log^2 n/\epsilon)$	Near $(3/2)$-approximation for **diameter**
$\tilde{O}(n^{1/6})$	Exact weighted undirected **SSSP**

Open Questions

- Can we get a $(2+\epsilon)$-approximation for weighted APSP?
- Can we get sub-polynomial algorithm for exact SSSP? Or directed SSSP?