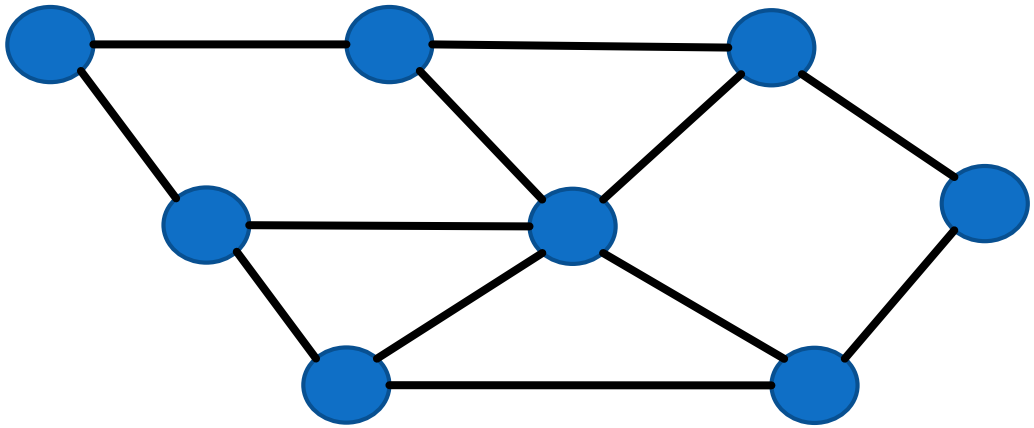
Distributed Approximation for Tree Augmentation

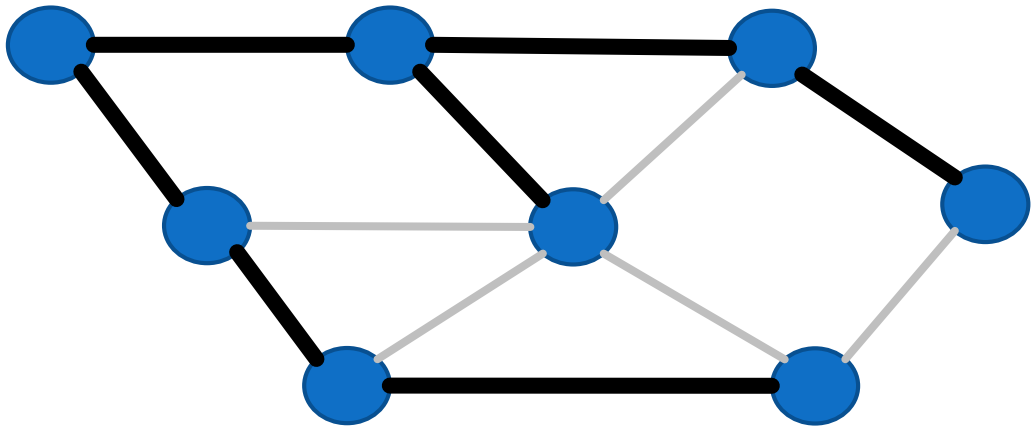
Michal Dory, Technion

Joint work with: Keren Censor-Hillel, Technion

Consider a Communication Network

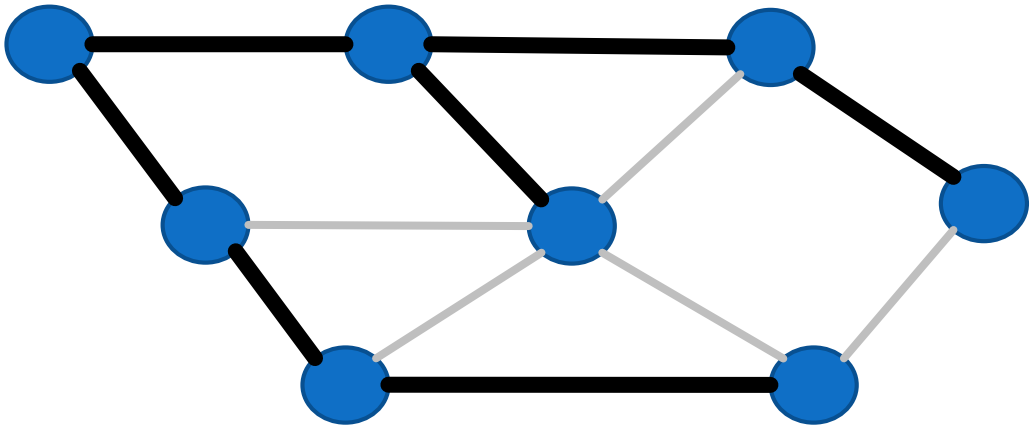


We can communicate over a spanning tree

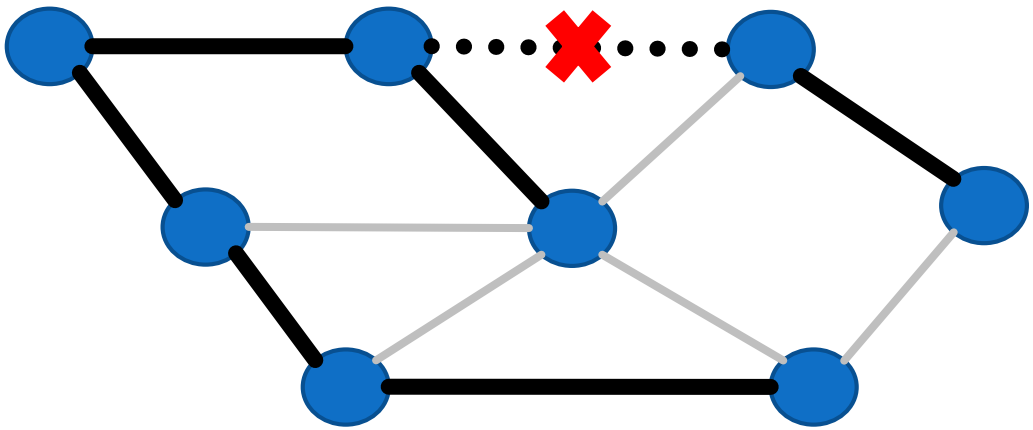


There are many constructions of minimum spanning trees (MST)

[Gallager, Humblet and Spira 83, Kutten and Peleg 95, Garay, Kutten and Peleg 98, Pandurangan, Robinson and Scquizzato 17, Elkin 17,...]



Trees cannot survive a link failure



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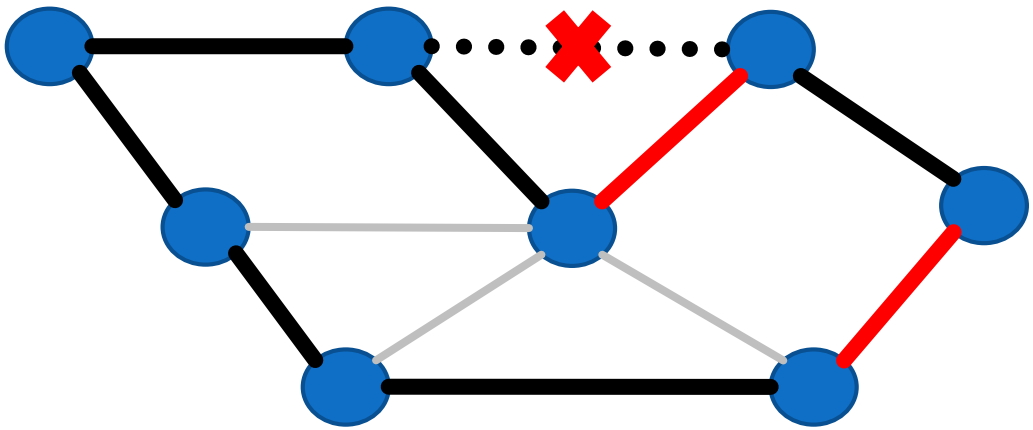
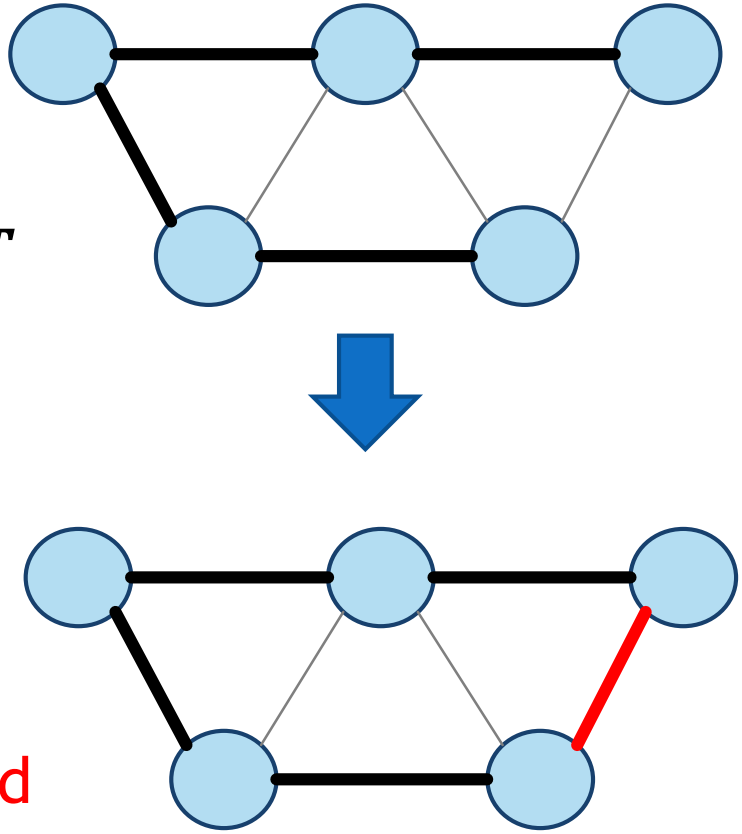


The Tree Augmentation Problem (TAP)

Input:
a graph G
a spanning tree $\boldsymbol{T}$



Goal:
augment $\boldsymbol{T}$ to be
2-edge-connected

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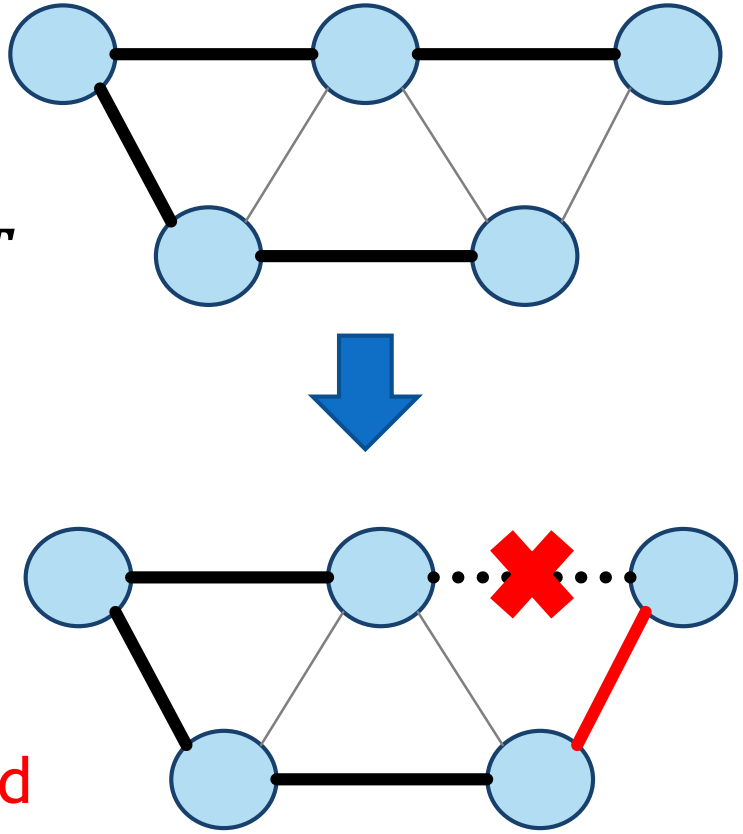

Goal:
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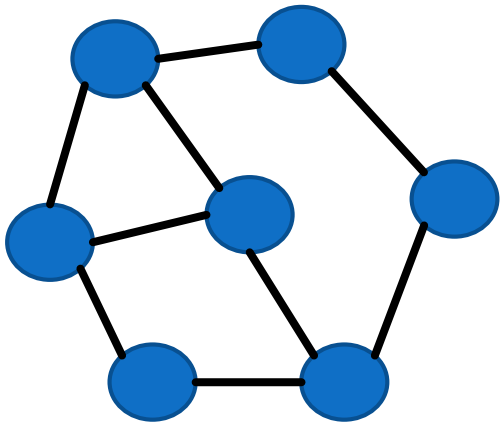
The Tree Augmentation Problem (TAP)

- A central problem in network design
- Has many **sequential** algorithms:

  [Frederickson and JáJá 81, Khuller and Thurimella 93, Goemans et al. 94, Jain 01, Kortsarz and Nutov 16, Adjiashvili 17…].
- Goal: solve TAP in the **distributed** setting.

The CONGEST model

- Communication network with n processors
- Synchronous rounds
- Messages of $O(\log n)$ bits
- Time = number of rounds
- The input and output are local



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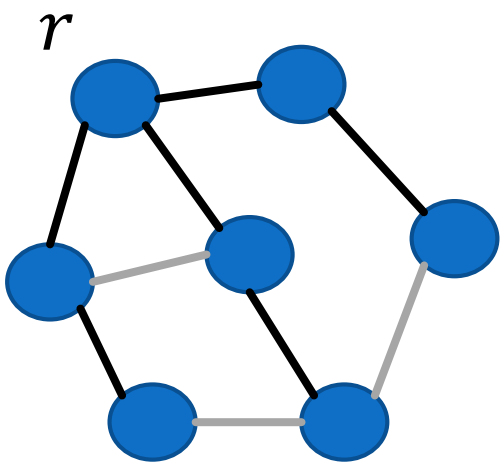
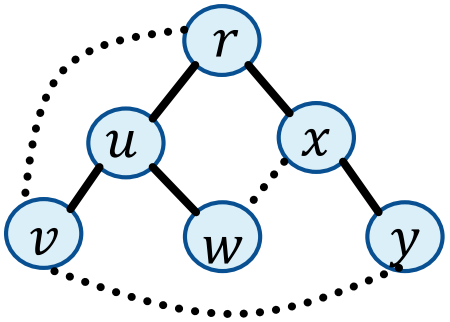


Sequential Approximation for TAP
[Khuller and Thurimella 93]

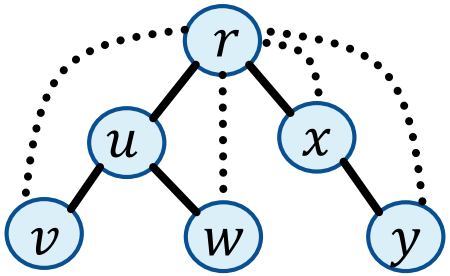
- Builds a new **virtual graph** G'
- Finds a **directed MST** in G'
- This gives a 2-approximation in G

Distributed Approximation for TAP

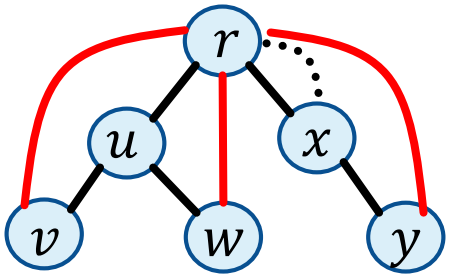
The input graph G



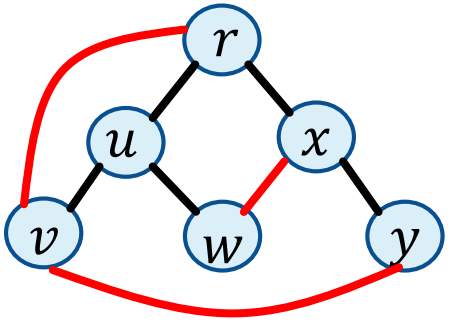
A virtual graph G'



Optimal solution in G'



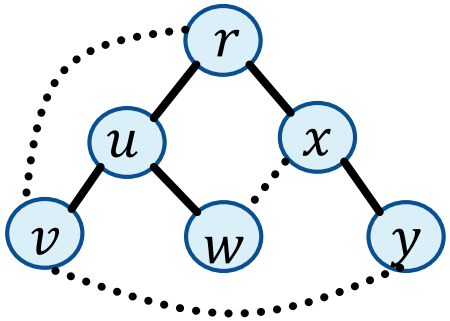
2-approximation in G



The Graph G'

- All the non-tree edges in G' are between ancestors to descendants:

The input graph G



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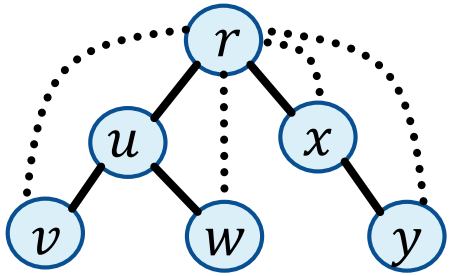

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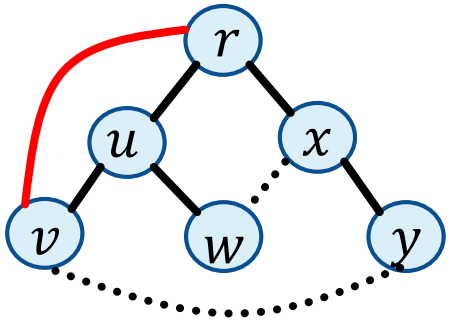

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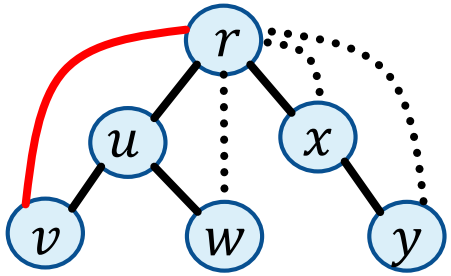

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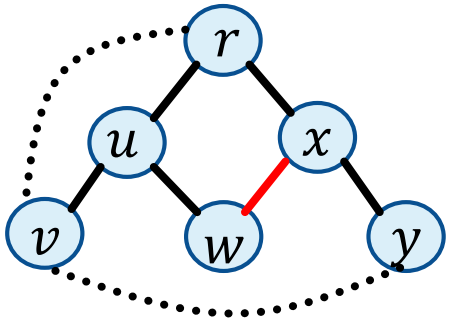

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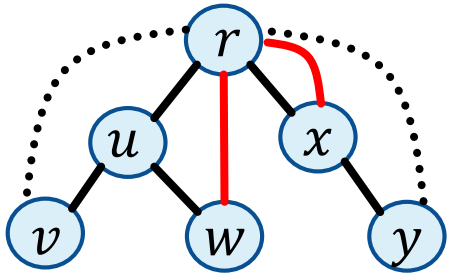
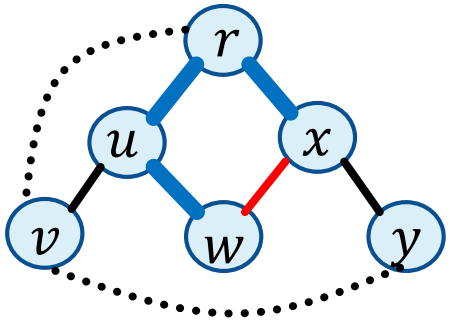


The Graph G'

- The corresponding edges in G' *cover* exactly the same tree edges:

The input graph G



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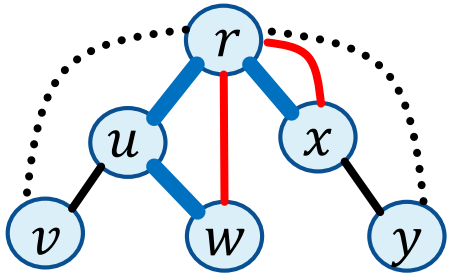
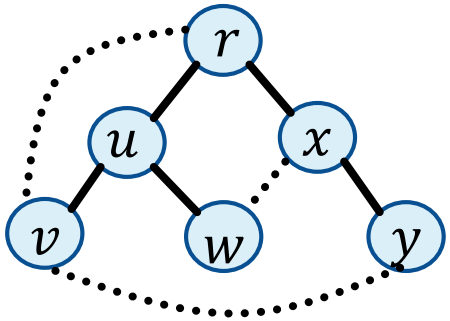


The Graph G'

- We can build G' in $O(h)$ rounds using LCA labels of $O(\log n)$ bits [Alstrup et al. 2004]

The input graph G



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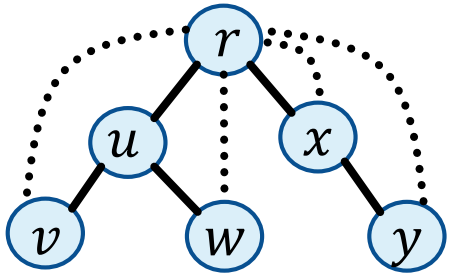
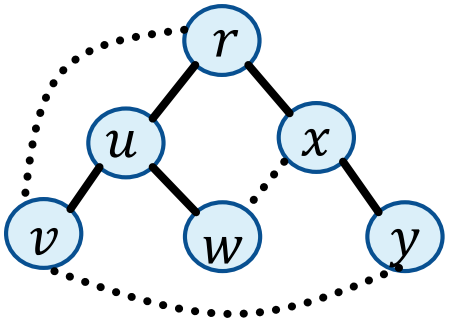


Distributed Approximation for TAP

The input graph G



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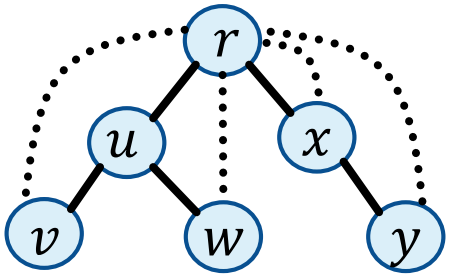
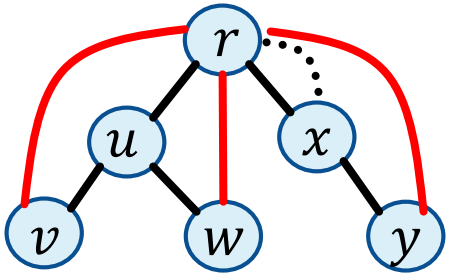
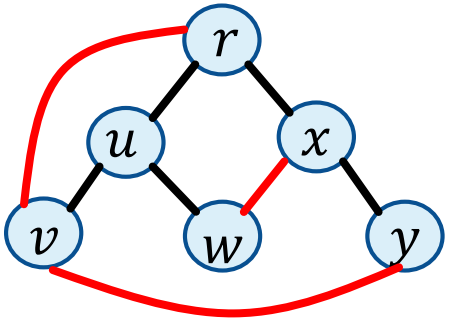


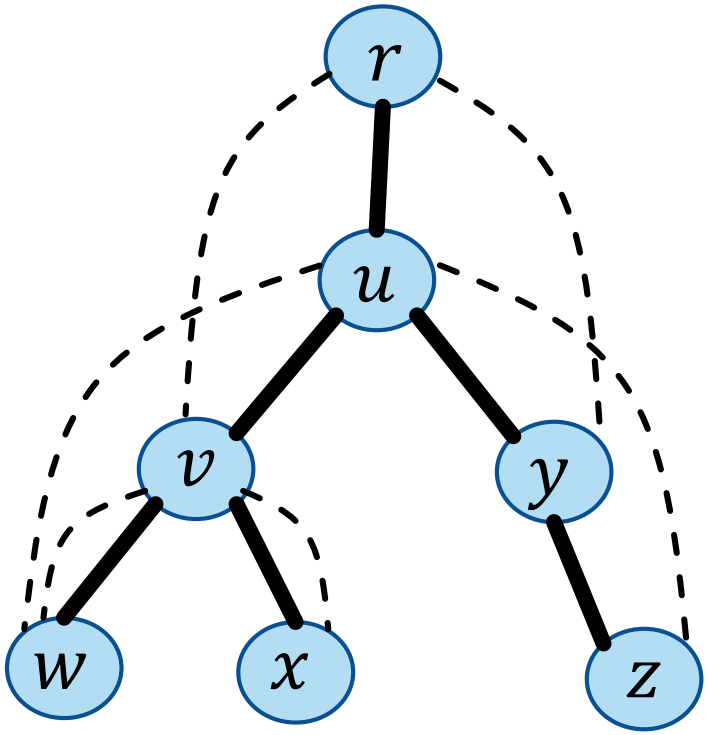
Optimal solution in G'



2-approximation in G



Finding an Optimal Augmentation in G'



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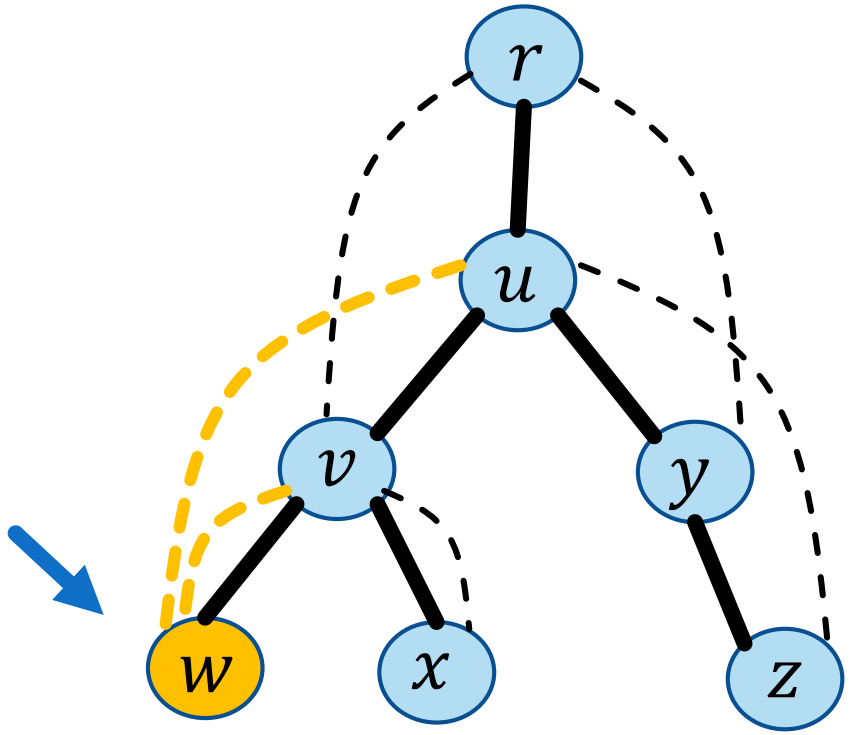

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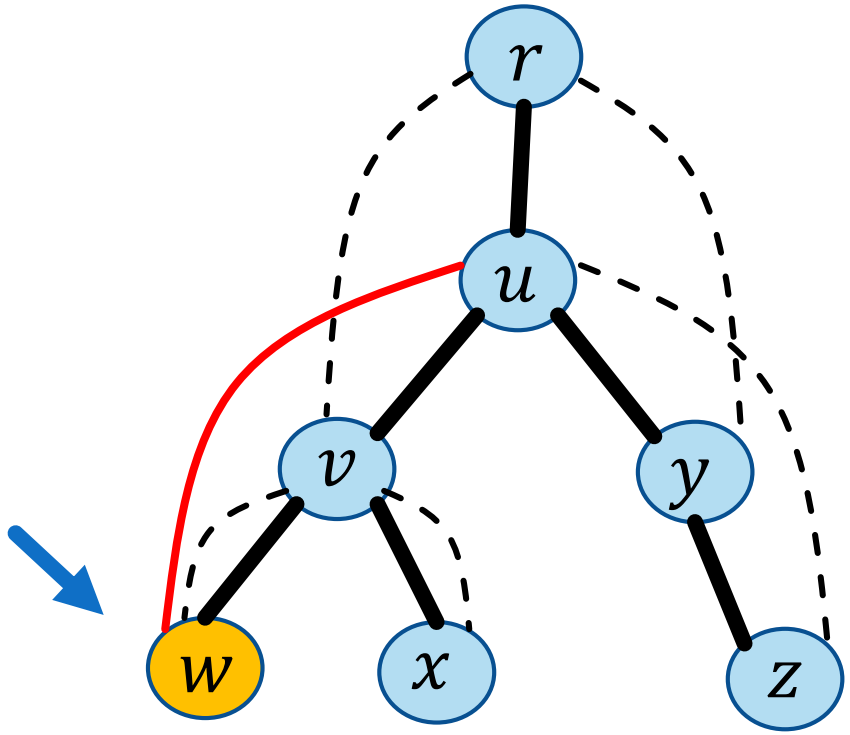

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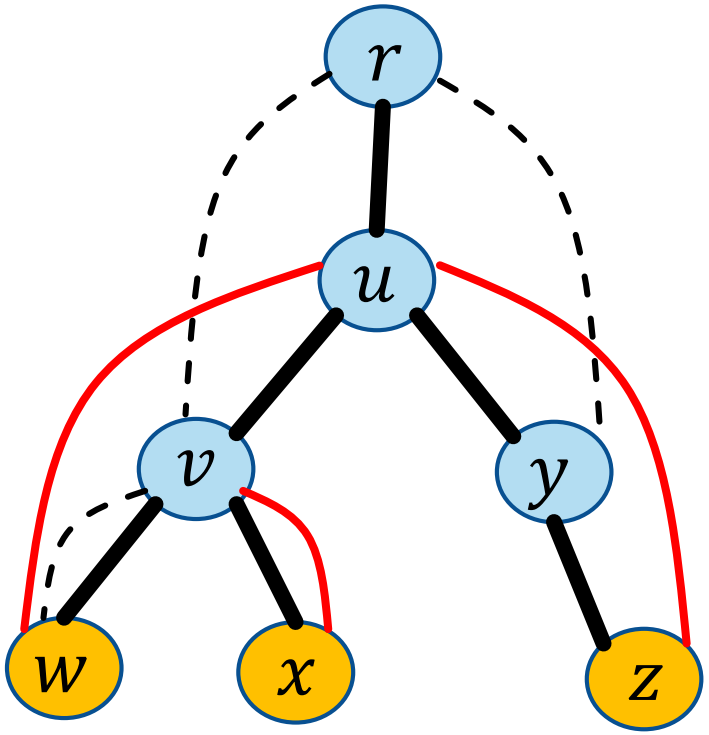

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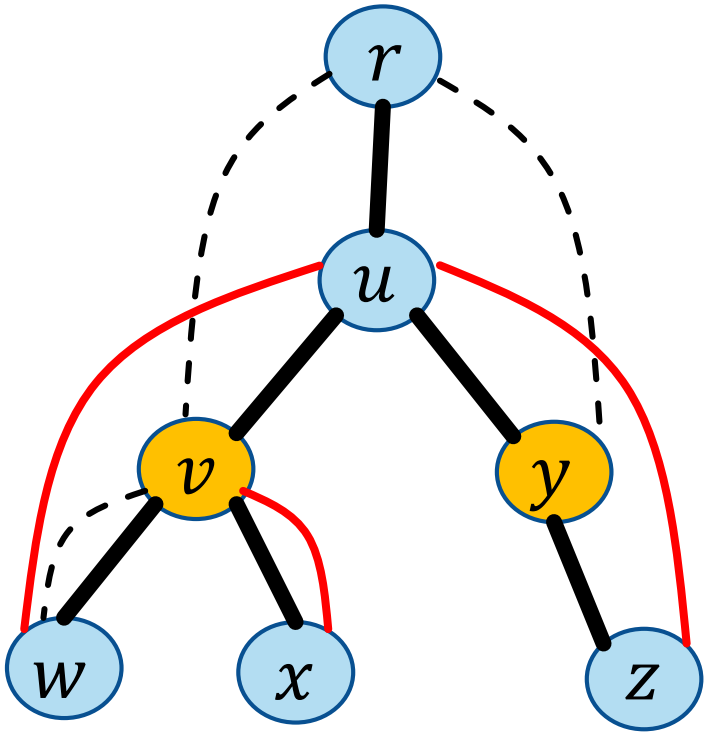

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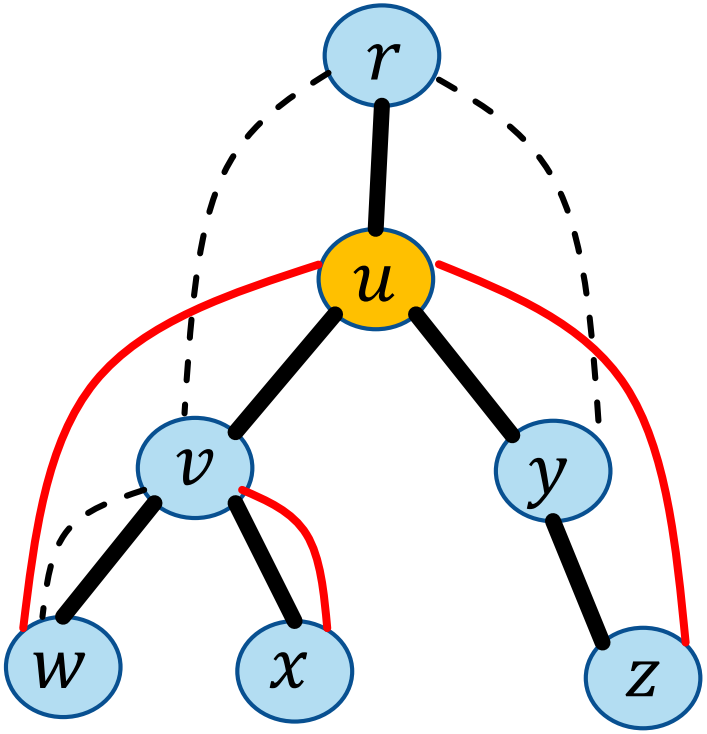

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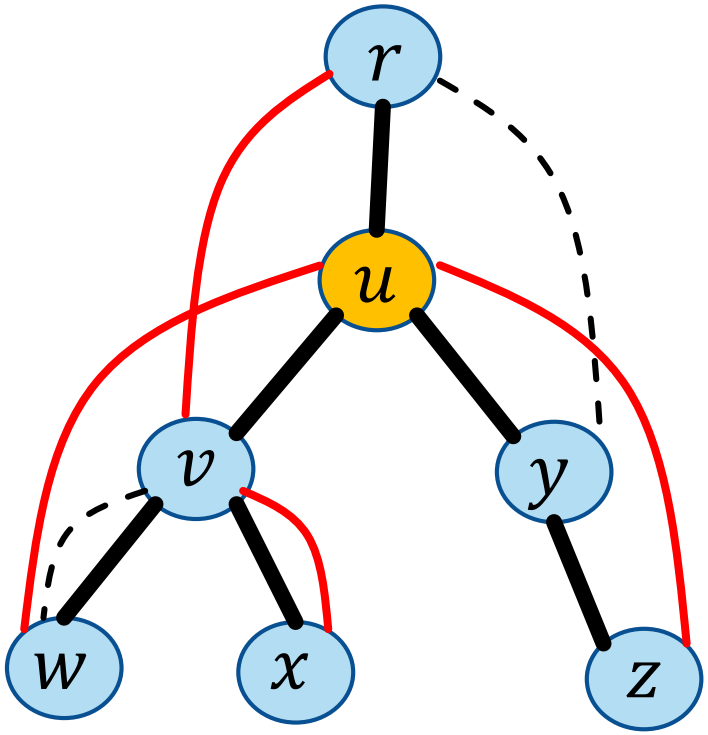

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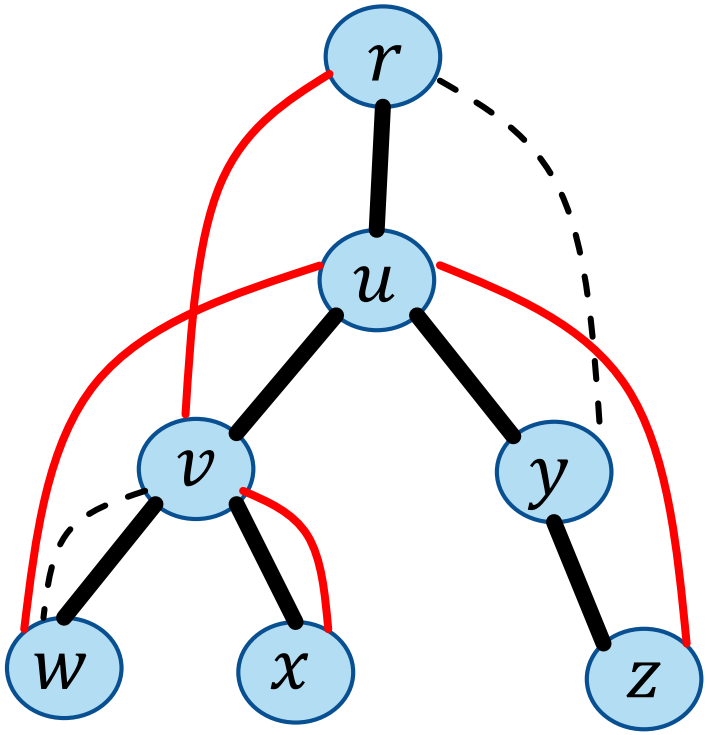


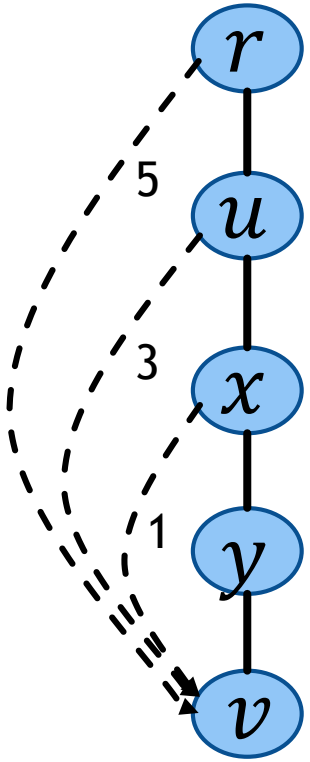
Conclusion

- There is a **2-approximation** for *unweighted* TAP in $\boldsymbol{O(h)}$ rounds, where h = height of T.
- What about the weighted case?

Weighted TAP

Problem:

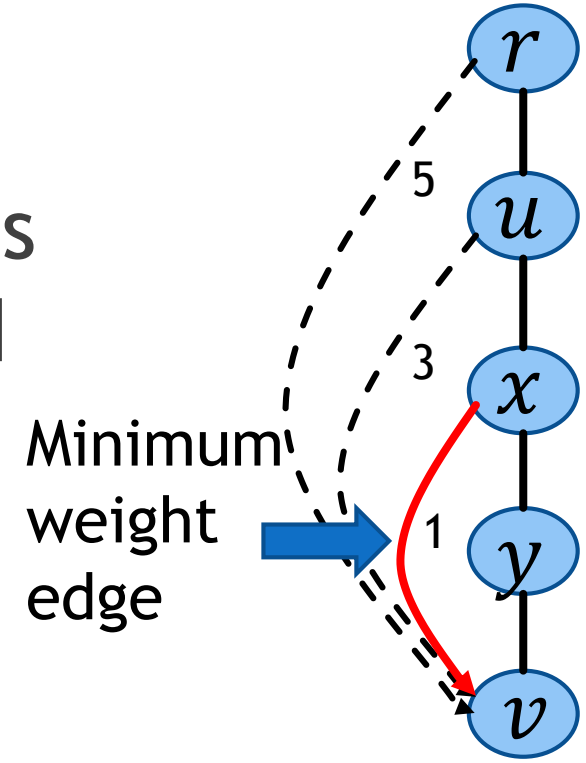
how to compare edges?



Weighted TAP

Solution:

sending edges with reduced weights



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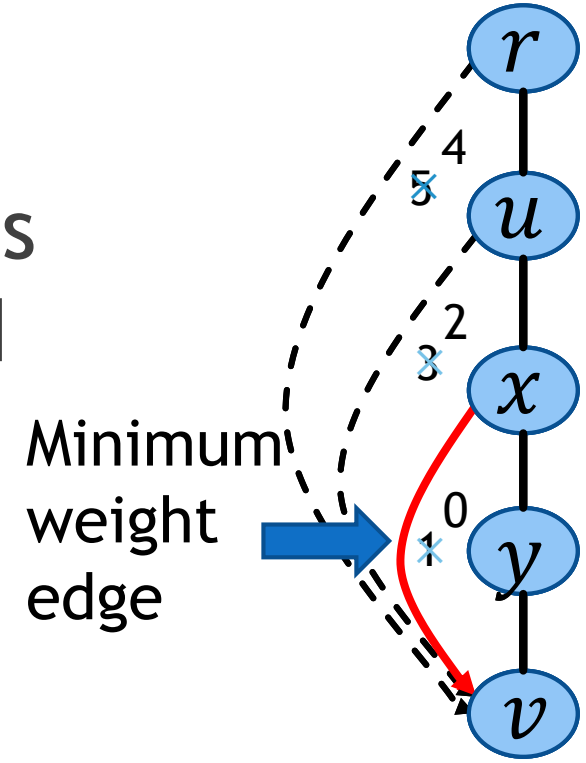
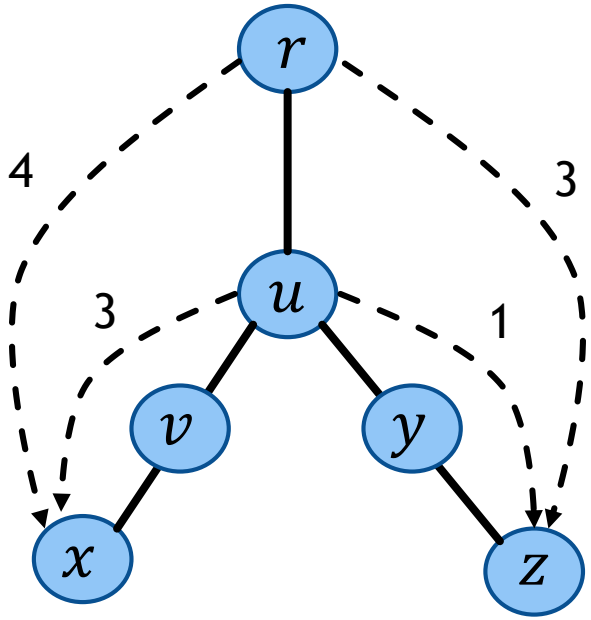


Example



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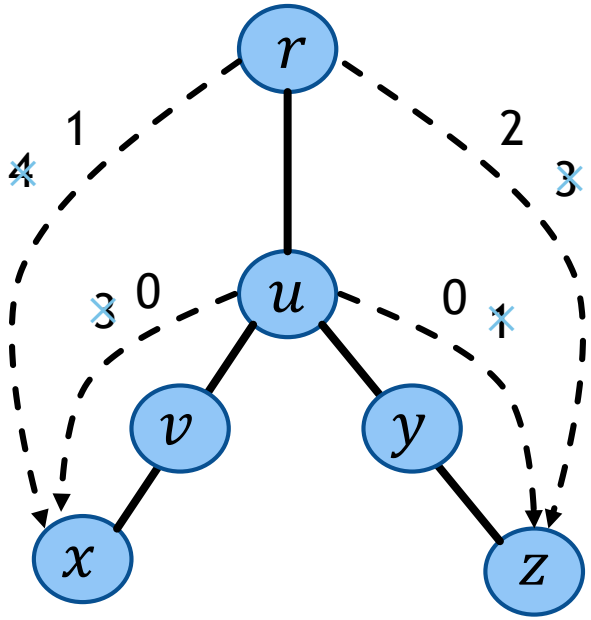

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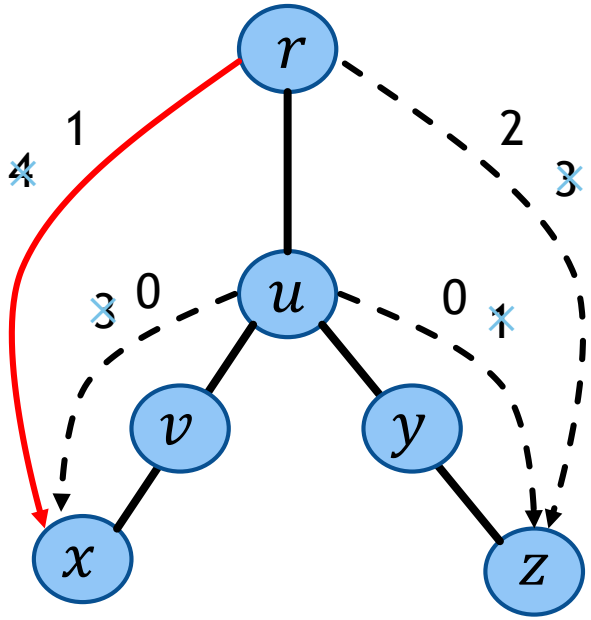

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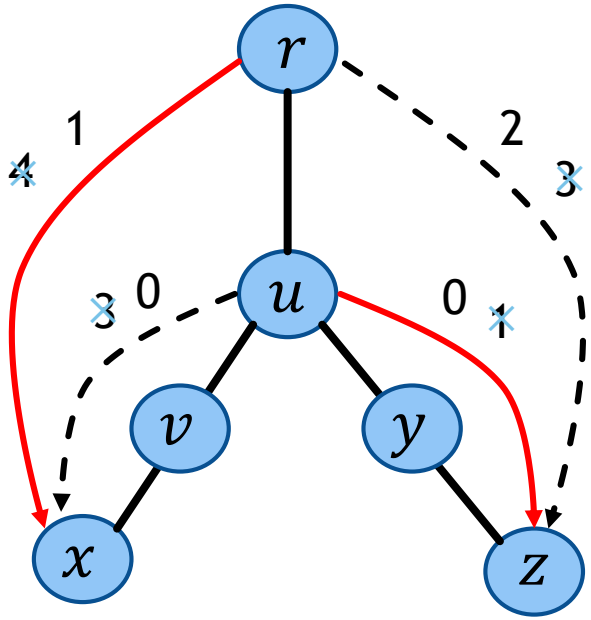

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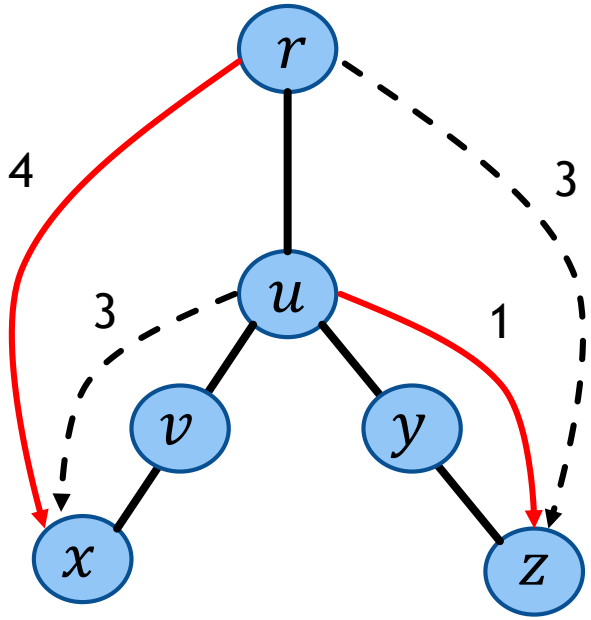
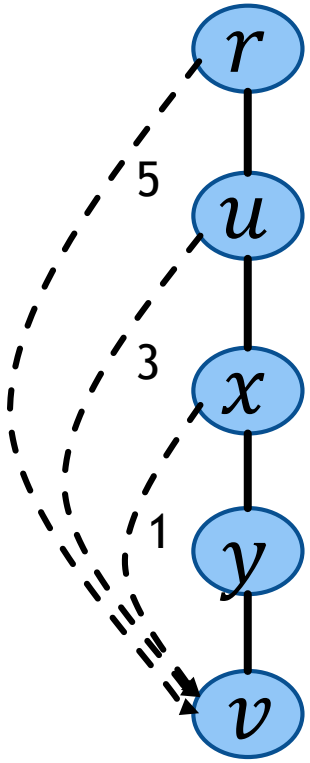


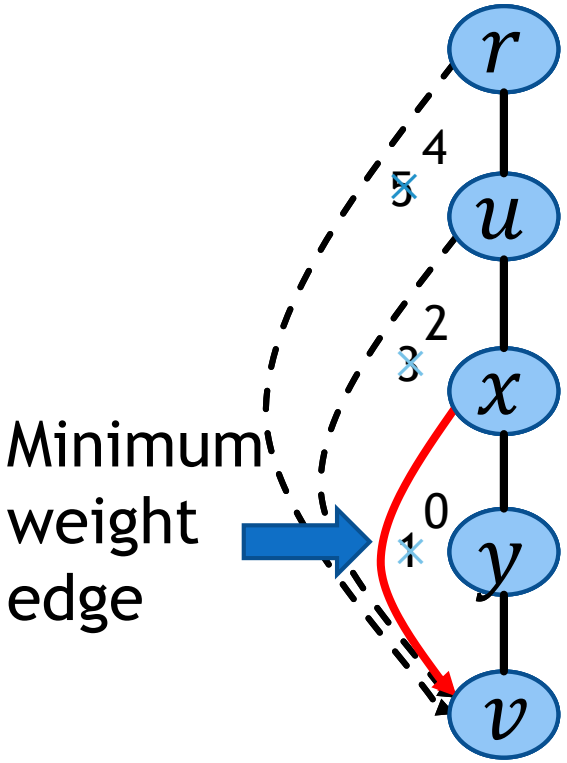
The Algorithm

Each vertex v computes the weights of the minimum weight edges that cover the path from v to each of its ancestors (at most h edges).



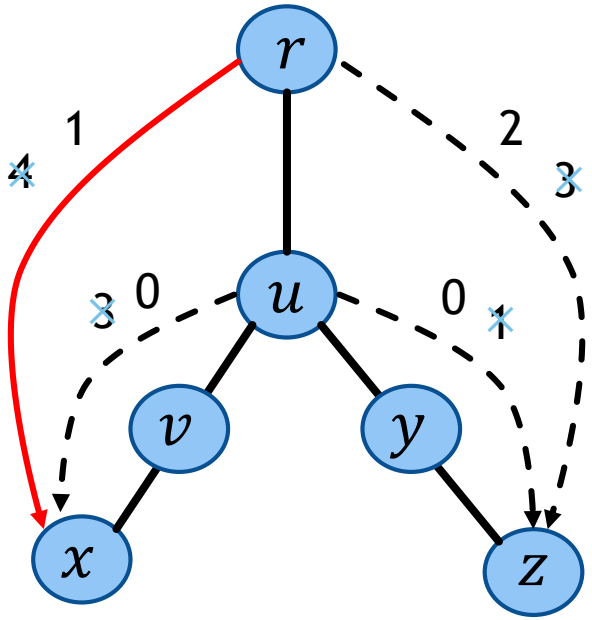
The Algorithm

It sends to its parent these edges after reducing from them the weight min_v of the minimum weight edge covering $\{v, p(v)\}$.



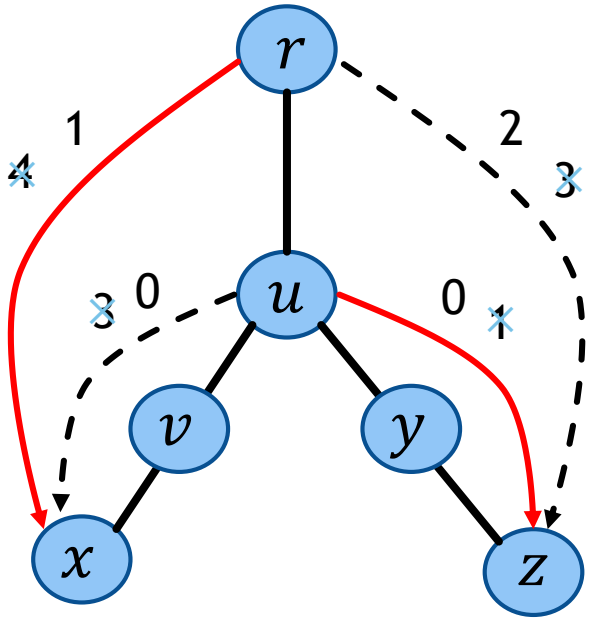
The Algorithm

At the end each child v of r adds to the augmentation the edge having weight min_v.



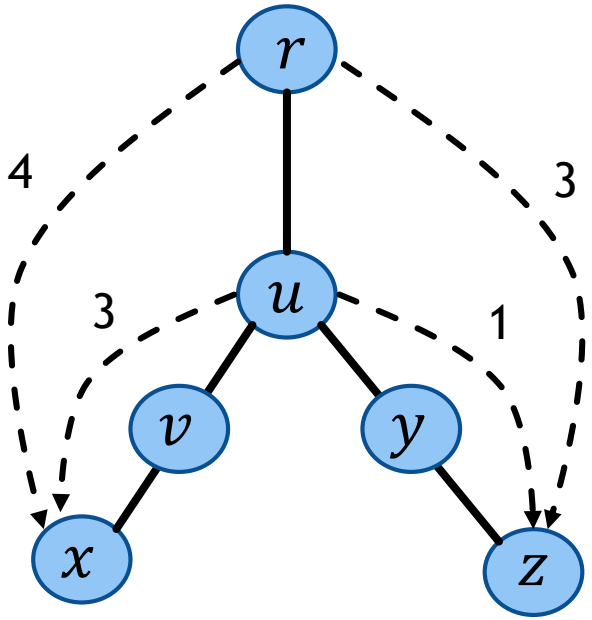
The Algorithm

A vertex v such that $\{v, p(v)\}$ is still not covered, cover it by the edge having weight min_v.



Correctness Proof

We give to a tree edge $\{v, p(v)\}$ the cost min_v.



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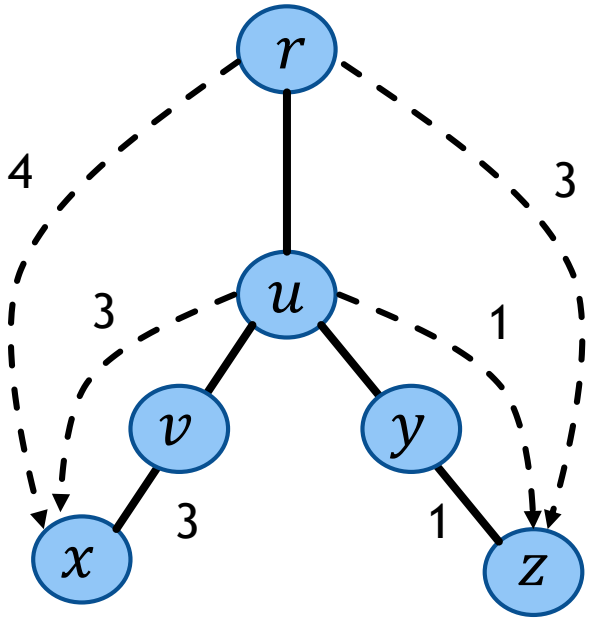

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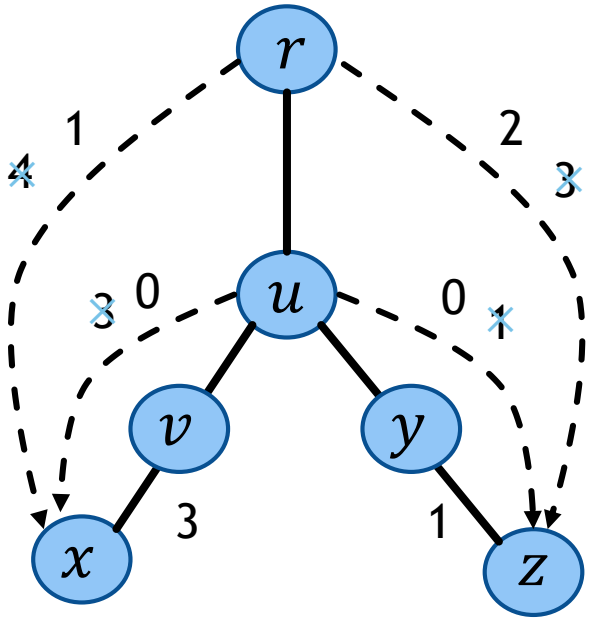

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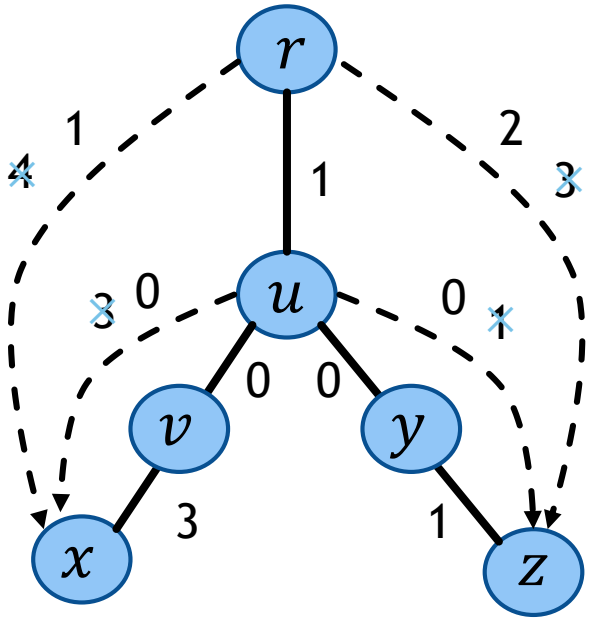

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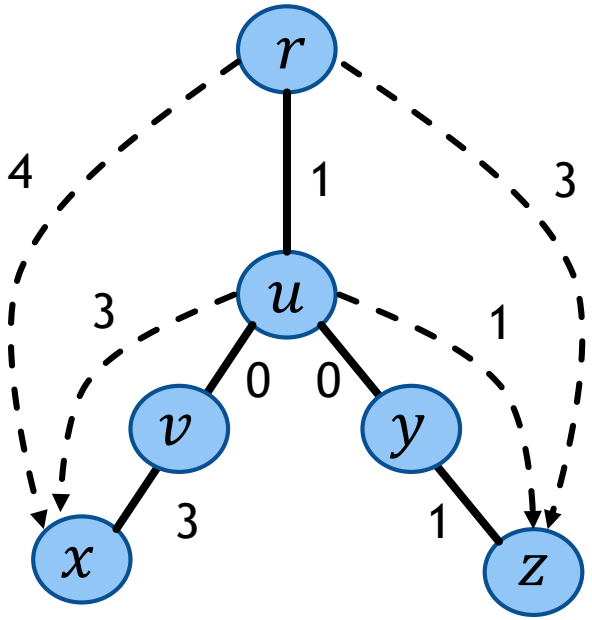
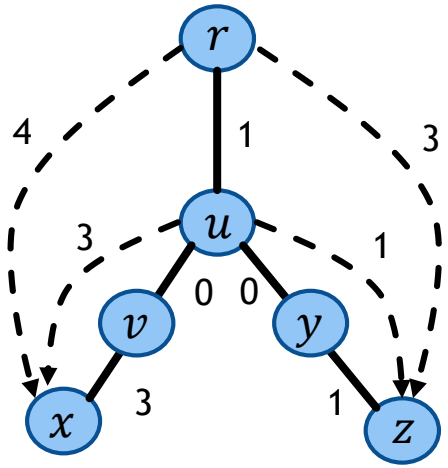


Correctness Proof

The sum of the costs is equal to the cost of the solution obtained in the algorithm and to the cost of an optimal solution.

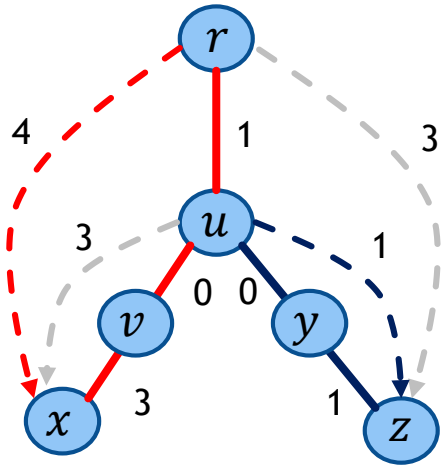


Correctness Proof

- For each edge $e \in A$ we assign a path P_e such that

  $w(e) = \sum_{t \in P_e} c(t)$.

- The paths P_e are disjoint and include all tree edges.



Conclusion

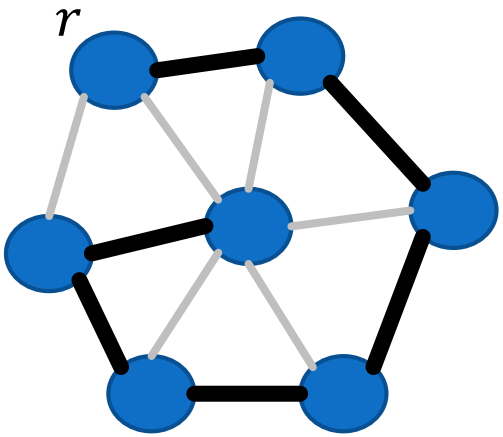
- There is a **2-approximation** for *weighted* TAP in $\boldsymbol{O(h)}$ rounds, where h = height of T.

Is This Optimal?

- TAP is a global problem which requires $\boldsymbol{\Omega(D)}$ rounds, where D = diameter of G
- If $h = O(D)$ our algorithms are optimal up to a constant factor

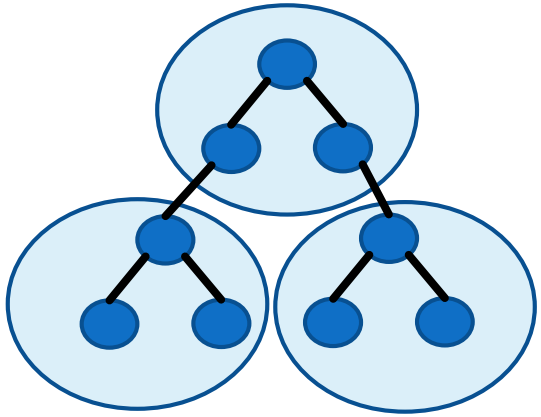
What about the case $h = \omega(D)$?

- We show an $\boldsymbol{\Omega(h)}$ lower bound for *weighted* TAP when $D \approx \log n$, $h \approx \sqrt{n}$



What about the case $h = \omega(D)$?

- We show a **4-approximation** for unweighted TAP in $\boldsymbol{\tilde{O}(\sqrt{n} + D)}$ rounds

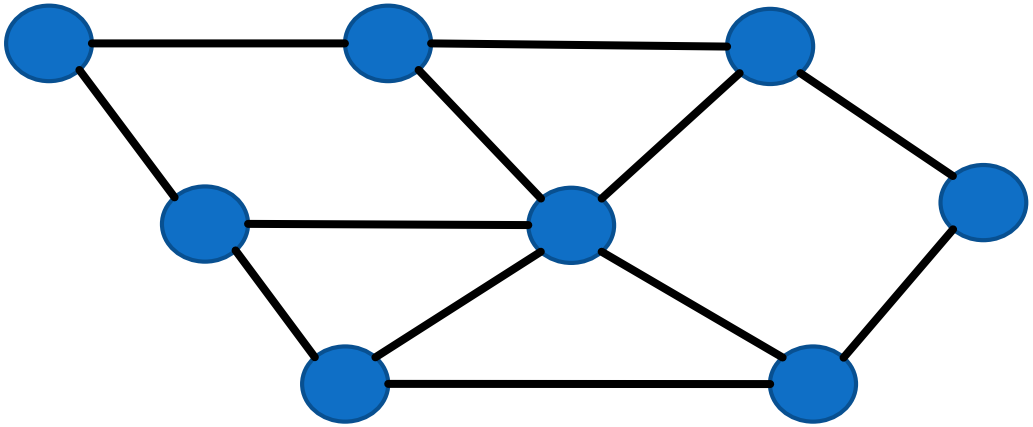


Main Results

- **2-approximation** for unweighted or weighted TAP in $\boldsymbol{O(h)}$ rounds, where h = height of T
- **4-approximation** for unweighted TAP in $\boldsymbol{\tilde{O}(\sqrt{n} + D)}$ rounds, where D = diameter of G
- $\boldsymbol{\Omega(D)}$ lower bound
- $\boldsymbol{\Omega(h)}$ lower bound for weighted TAP when $D \approx \log n$, $h \approx \sqrt{n}$

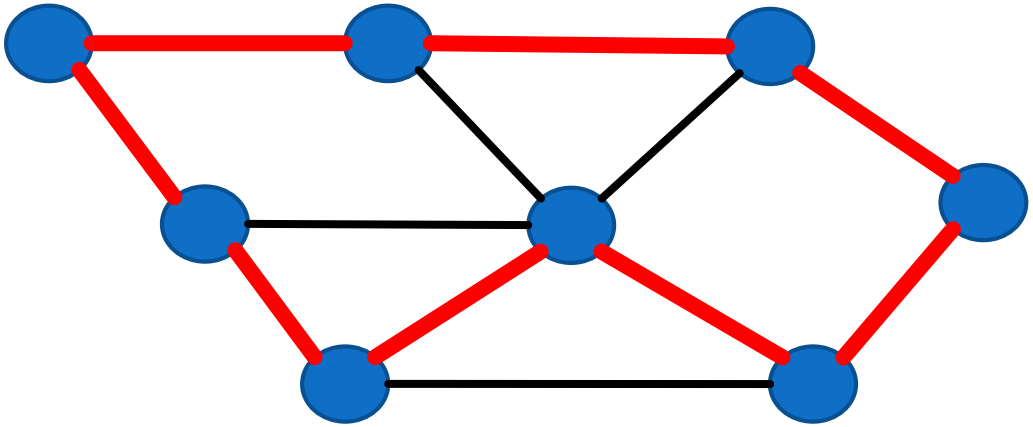
Application: minimum size 2-Edge-Connected Subgraph (2-ECSS)

- Goal: find the minimum size 2-ECSS
- No spanning tree is given



Application: minimum size 2-Edge-Connected Subgraph (2-ECSS)

- Goal: find the minimum size 2-ECSS
- No spanning tree is given



Application: minimum size 2-Edge-Connected Subgraph (2-ECSS)

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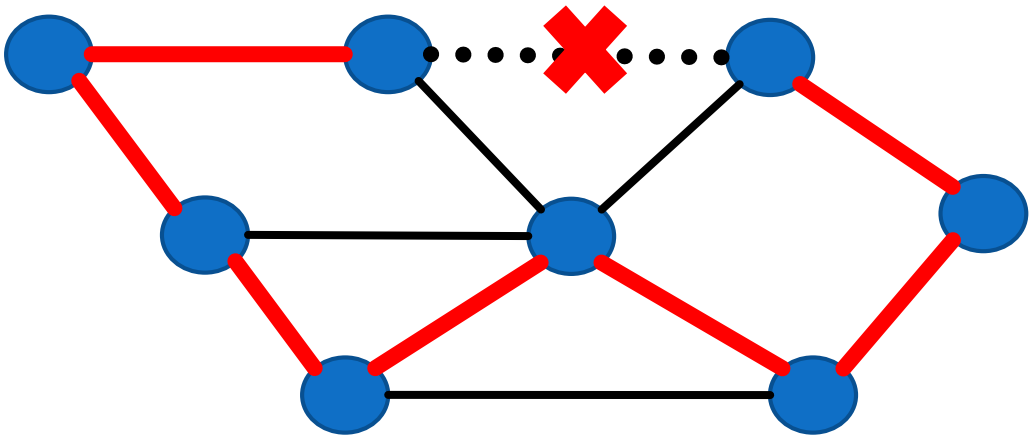

Application: minimum size 2-Edge-Connected Subgraph (2-ECSS)

Previous algorithms:

- $\frac{3}{2}$-approximation in $O(n)$ rounds [Krumke et al. 07]
- 2-approximation in $\tilde{O}(D+\sqrt{n})$ rounds [Thurimella 95]

Application: minimum size 2-Edge-Connected Subgraph (2-ECSS)

Previous algorithms:

- $\frac{3}{2}$-approximation in $O(n)$ rounds [Krumke et al. 07]
- 2-approximation in $\tilde{O}(D+\sqrt{n})$ rounds [Thurimella 95]

Using our TAP algorithm:

- 2-approximation in $O(D)$ rounds

Application: minimum ***weight*** 2-Edge-Connected Subgraph (2-ECSS)

Previous algorithms:

- 3-approximation in $O(n \log n)$ rounds

  [Krumke et al. 07]

Using our TAP algorithm:

- 3-approximation in $\tilde{O}(h_{MST} + \sqrt{n})$ rounds
- Lower bound of $\tilde{\Omega}(D + \sqrt{n})$ rounds for any polynomial approximation

Future Work

- Design efficient algorithms for *weighted* TAP and *weighted* 2-ECSS.
- Design **distributed** algorithms for additional **connectivity** problems:
  - Higher connectivity
  - Vertex connectivity

Can we improve the round complexity?

- We show an $\tilde{O}(D + \sqrt{n})$-round $O(\log n)$-approximation for weighted TAP and weighted 2-ECSS.

The Algorithm

- TAP is a set cover problem where the goal is to cover all tree edges by non-tree edges.
- The cost-effectiveness of a non-tree edge e is $\frac{C_e}{w(e)}$ where C_e is the number of uncovered tree edges covered by e.

Sequential Greedy Algorithm

- At each step, add to the augmentation the edge with maximal cost-effectiveness.
- Continue until all the tree edges are covered.

Distributed Algorithm

- We need a mechanism for symmetry breaking.
- We need to do many global computations in parallel.

The Algorithm

- At each step, find all the edges with maximal cost-effectiveness, they are the *candidates*.
- Each candidate edge chooses a random number $r \in [0,1]$.
- Each uncovered tree edge votes to the first candidate that covers it.
- An edge is added to the augmentation if it gets at least $\frac{1}{8}$ of the votes of the tree edges it covers.
- We continue until all tree edges are covered.

The Algorithm

- We need to do many global computations in parallel: computing cost-effectiveness, computing the number of votes…
- To achieve this, we decompose the tree into fragments, following a decomposition presented for solving the fault-tolerant MST problem [Ghaffari and Parter, 2016].

Higher Connectivity

- What about minimum k-ECSS?
- The same basic approach may be useful also for k-ECSS.
- However, now the cost-effectiveness of an edge depends on the number of cuts it covers.