Distributed Spanner Approximation

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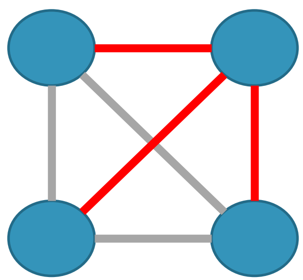
Spanners

A $\boldsymbol{k}$**-spanner** of a graph G is a subgraph of G that preserves distances up to a multiplicative factor of k.



Spanners

- There are many constructions which give a global guarantee on the size of the spanner:

  $(2k-1)$-spanners with $O(n^{1+1/k})$ edges

- This is optimal in the worst case assuming Erdős's girth conjecture.

Spanner Approximation

- What about approximating the minimum k-spanner?

$$\text{Number of edges} \leq \alpha \cdot OPT$$

- There are graphs where any 2-spanner has $\Omega(n^2)$ edges, this is also true for k-spanners in directed graphs.

In the sequential setting:

- 2-spanner: $O\left(\log\frac{|E|}{|V|}\right)$-approximation [Kortsarz and Peleg 1994]
- Directed k-spanner: $O(\sqrt{n}\ \log n)$-approximation [Berman, Bhattacharyya, Makarychev, Raskhodnikova and Yaroslavtsev 2013]

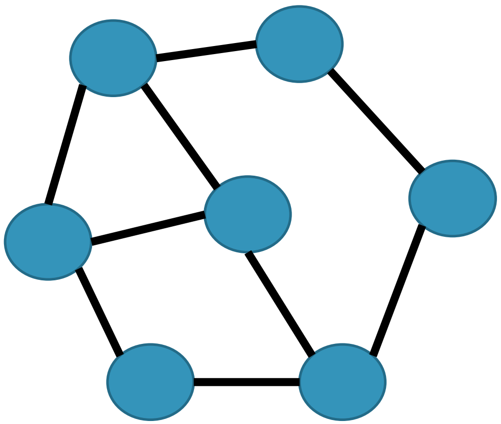
Hardness Results:

- 2-spanner: $\Omega(\log n)$ [Kortsarz 2001]
- Directed k-spanner: $\Omega(2^{(\log^{1-\varepsilon} n)})$[Elkin and Peleg 2007]
- Undirected k-spanner: $\Omega(2^{(\log^{1-\varepsilon} n)/k})$ [Dinitz, Kortsarz and Raz 2016]

The Distributed Models

n vertices exchange messages in **synchronous** rounds

The model	Message size
LOCAL	unbounded
CONGEST	$\Theta(\log n)$ bits



In the LOCAL model

2-spanners:

Approximation	Number of rounds	
$O(\log n)$	$O(\log n)$	[Dinitz and Krauthgamer, 2011]

Directed k-spanners:

Approximation	Number of rounds	
$O(\sqrt{n}\ \log n)$	$O(k \log n)$	[Dinitz and Nazari, 2017]
$O(n^{\epsilon})$	constant	[Barenboim, Elkin and Gavoille, 2016]
$(1+\epsilon)$	$O(poly(\log n/\epsilon))$	Our Results

In the CONGEST model

Undirected $(2k-1)$-spanners:

There are global constructions of spanners with $O(n^{1+1/k})$ edges

Approximation	Number of rounds	
$O(n^{1/k})$	k	[Elkin and Neiman, 2017]

Spanner Approximation

- Can we give efficient approximations also in the **CONGEST** model?

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Approximating k-spanners in **directed** or **weighted** graphs is hard in the CONGEST model.

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Approximating k-spanners in **directed** or **weighted** graphs is hard in the CONGEST model.

This gives a strict **separation** between:

- The **CONGEST** and **LOCAL** models
- The undirected and directed variants

Hardness of Approximation

Directed k-spanner for $k \geq 5$:

- Randomized algorithms - $\widetilde{\Omega}(\sqrt{n/\alpha})$ rounds for an α-approximation.
- Deterministic algorithms - $\widetilde{\Omega}(n/\sqrt{\alpha})$

Weighted k-spanner for $k \geq 4$:

- Directed graphs - $\widetilde{\Omega}(n)$
- Undirected graphs - $\widetilde{\Omega}(n/k)$

How to show the results?

- We show reductions from problems in communication complexity.

0	1	1	0	0	1	1

0	0	1	1	1	0	1

How to show the results?

- Learning if two input strings of size N are disjoint requires exchanging $\Omega(N)$ bits.

0	1	**1**	0	0	1	**1**
0	0	**1**	1	1	0	**1**

How to show the results?

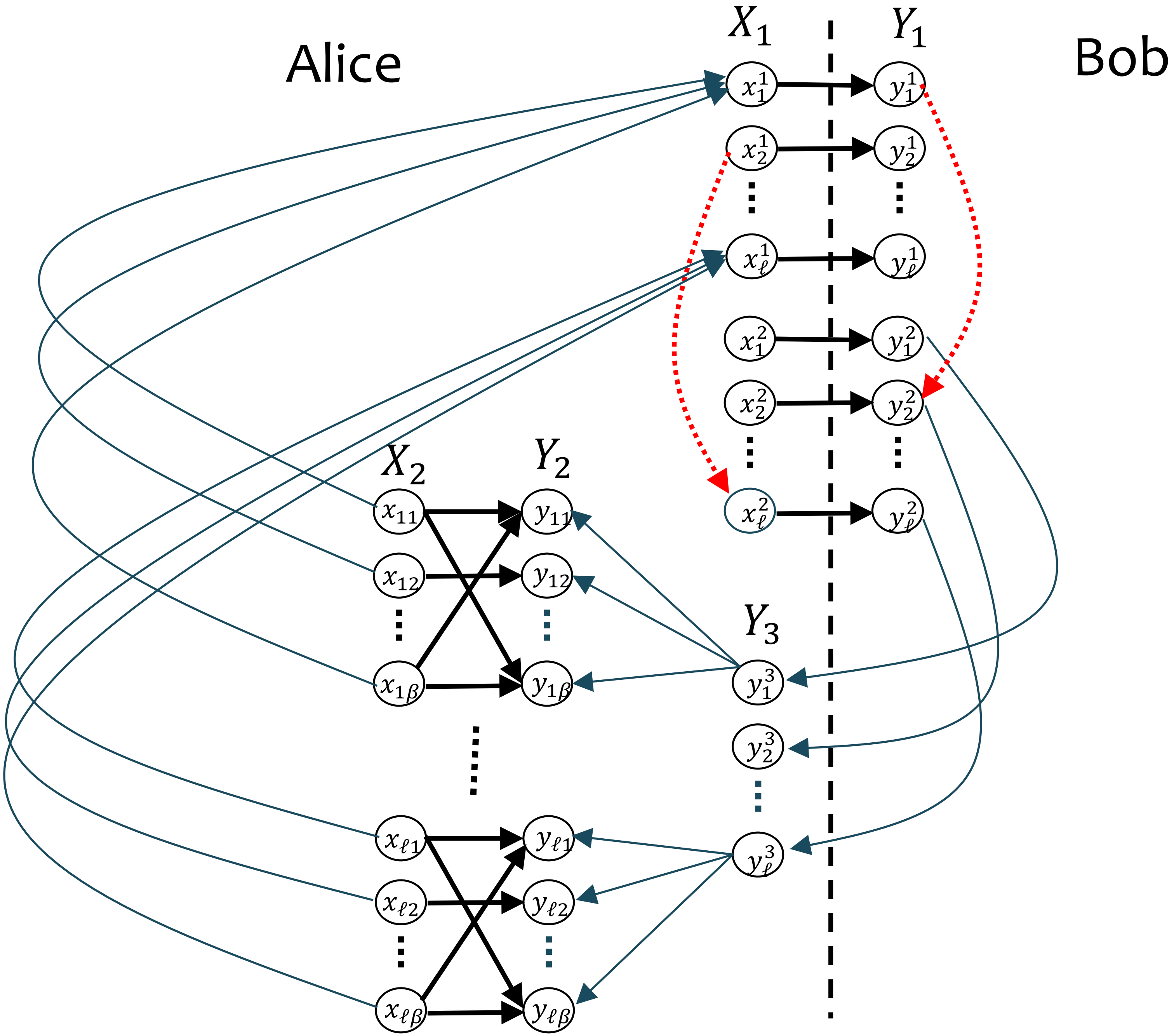
- **The goal:** create a graph G that depends on the inputs of Alice and Bob, such that

G has a sparse spanner $\Leftrightarrow$ the inputs satisfy some property

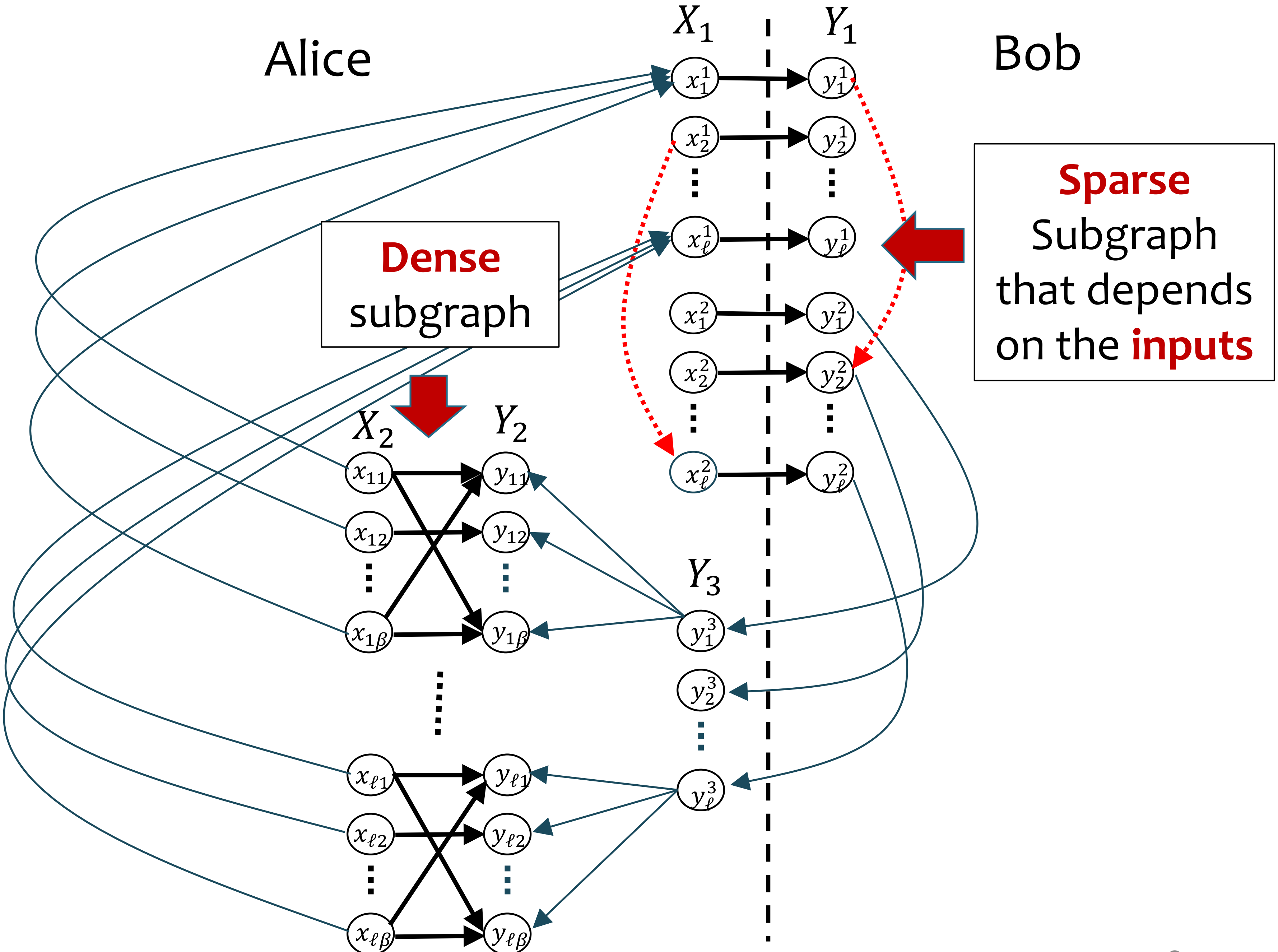
Creating gaps

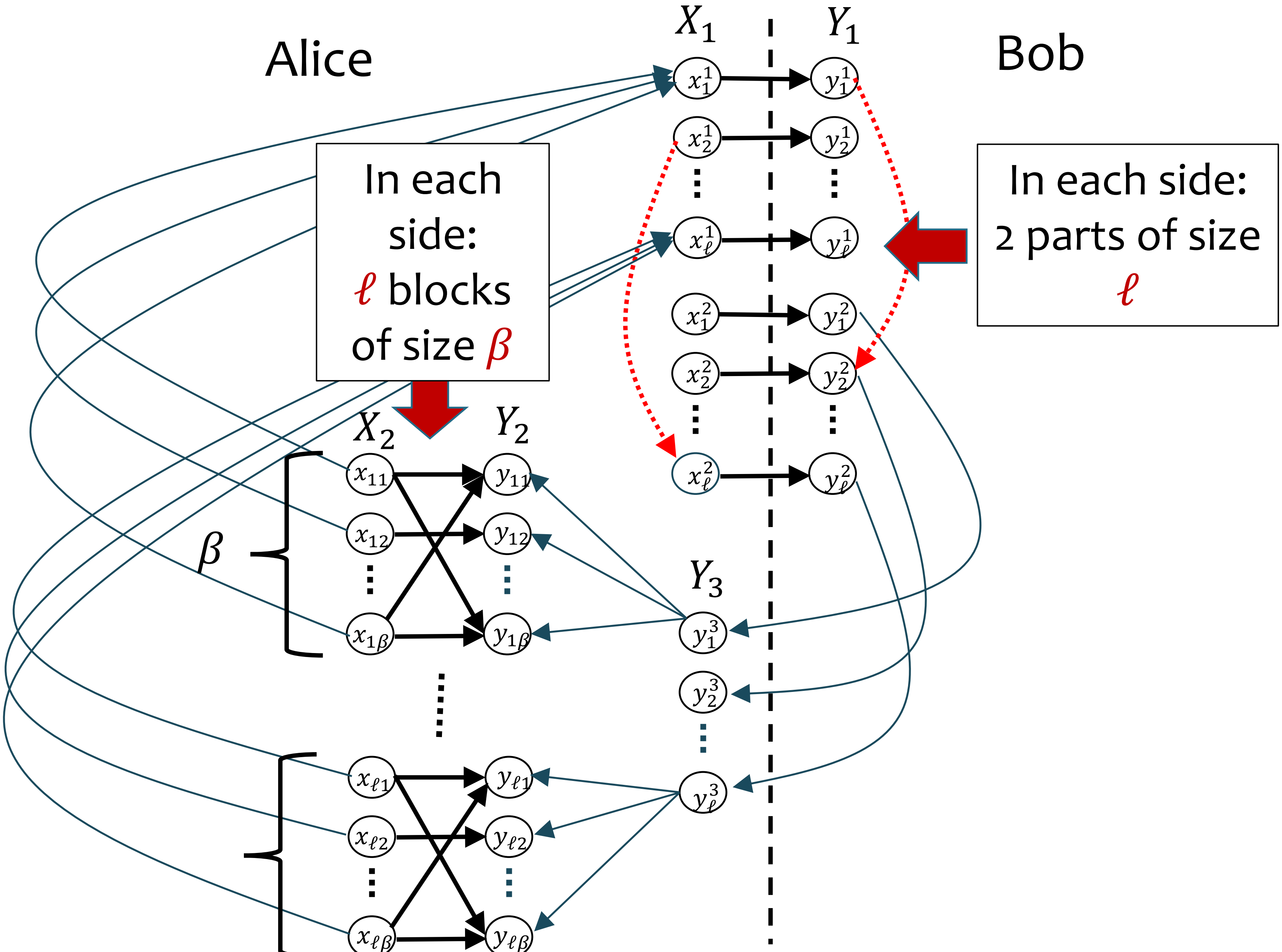
We show several approaches to **create gaps**:

- A construction where each input bit affects $\Omega(\alpha n)$ edges of the spanner
- Using the gap-disjointness problem
- Using the weights

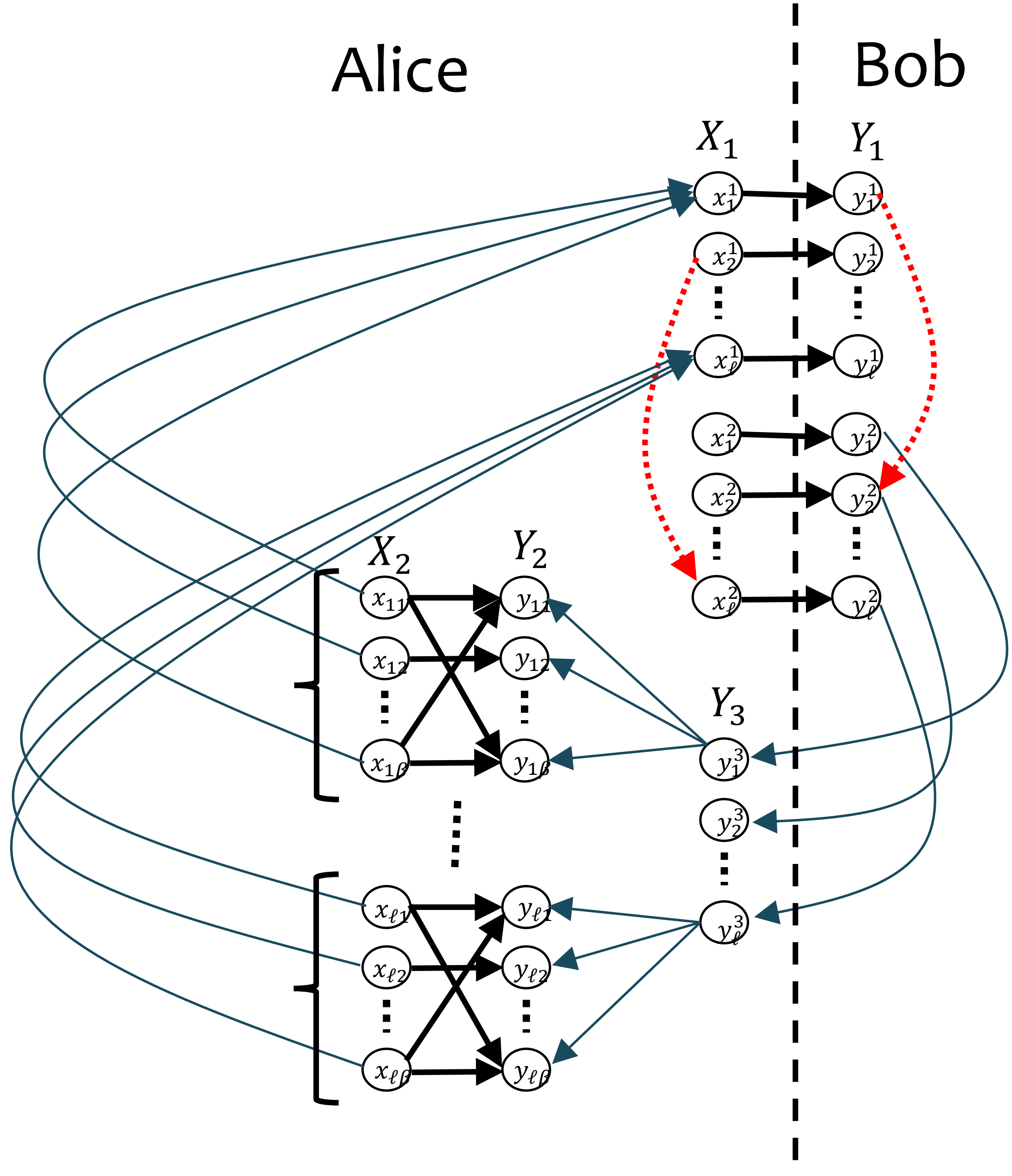


Alice
Bob
X_1
Y_1
x_1^1
y_1^1
x_2^1
y_2^1
x_ℓ^1
y_ℓ^1
x_1^2
y_1^2
x_2^2
y_2^2
x_ℓ^2
y_ℓ^2
X_2
Y_2
x_{11}
y_{11}
x_{12}
y_{12}
$x_{1\beta}$
$y_{1\beta}$
$x_{\ell 1}$
$y_{\ell 1}$
$x_{\ell 2}$
$y_{\ell 2}$
$x_{\ell\beta}$
$y_{\ell\beta}$
Y_3
y_1^3
y_2^3
y_ℓ^3


Alice
Bob
X_1
Y_1
X_2
Y_2
Y_3
Dense subgraph
Sparse Subgraph that depends on the inputs
x_1^1
x_2^1
x_ℓ^1
x_1^2
x_2^2
x_ℓ^2
y_1^1
y_2^1
y_ℓ^1
y_1^2
y_2^2
y_ℓ^2
x_{11}
x_{12}
$x_{1\beta}$
y_{11}
y_{12}
$y_{1\beta}$
$x_{\ell 1}$
$x_{\ell 2}$
$x_{\ell\beta}$
$y_{\ell 1}$
$y_{\ell 2}$
$y_{\ell\beta}$
y_1^3
y_2^3
y_ℓ^3


Alice
Bob
X_1
Y_1
x_1^1
x_2^1
x_ℓ^1
x_1^2
x_2^2
x_ℓ^2
y_1^1
y_2^1
y_ℓ^1
y_1^2
y_2^2
y_ℓ^2
In each side: ℓ blocks of size β
In each side: 2 parts of size ℓ
X_2
Y_2
Y_3
β
x_{11}
x_{12}
$x_{1\beta}$
$x_{\ell 1}$
$x_{\ell 2}$
$x_{\ell\beta}$
y_{11}
y_{12}
$y_{1\beta}$
$y_{\ell 1}$
$y_{\ell 2}$
$y_{\ell\beta}$
y_1^3
y_2^3
y_ℓ^3

Each block is connected to one vertex outside the block



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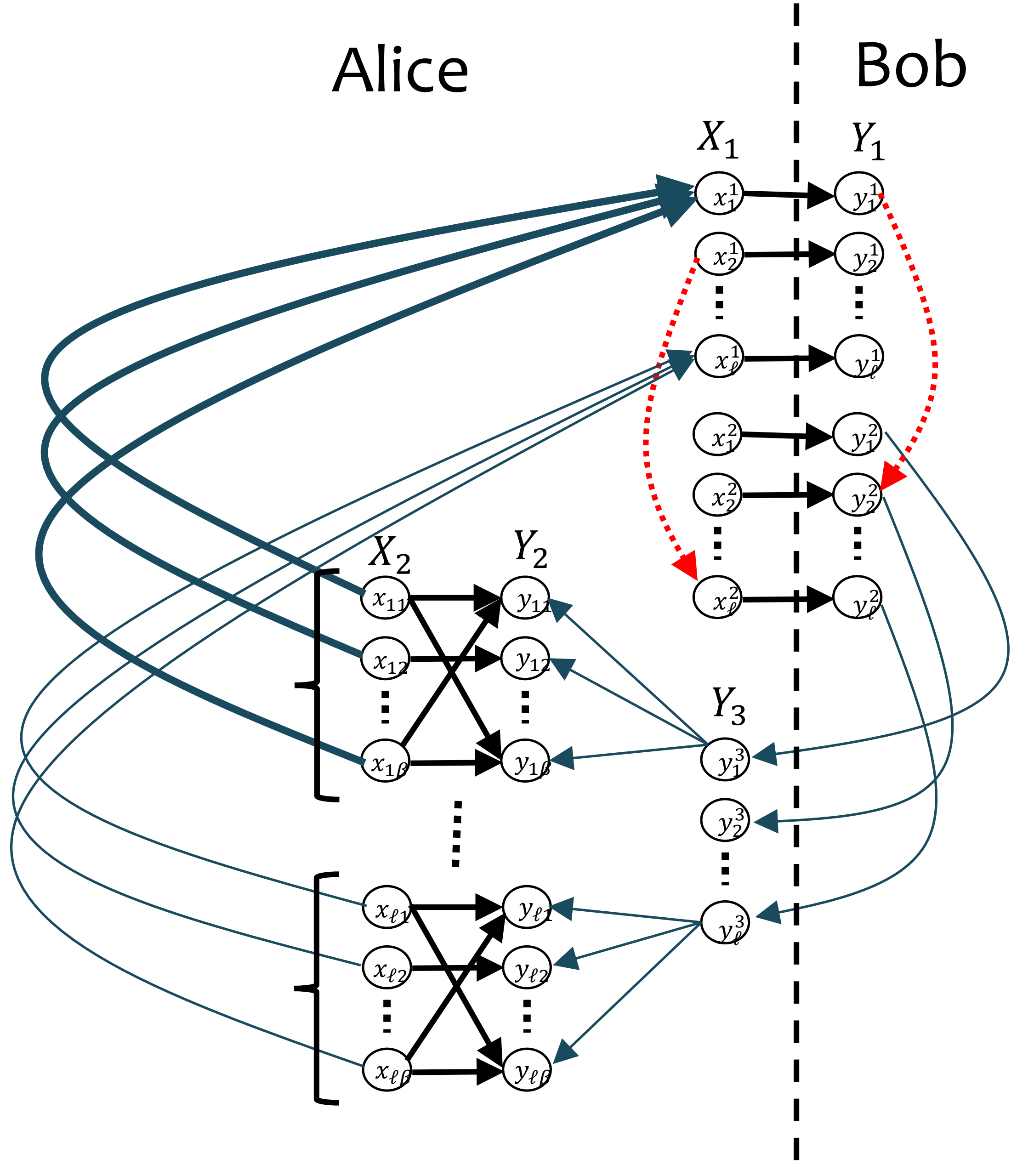

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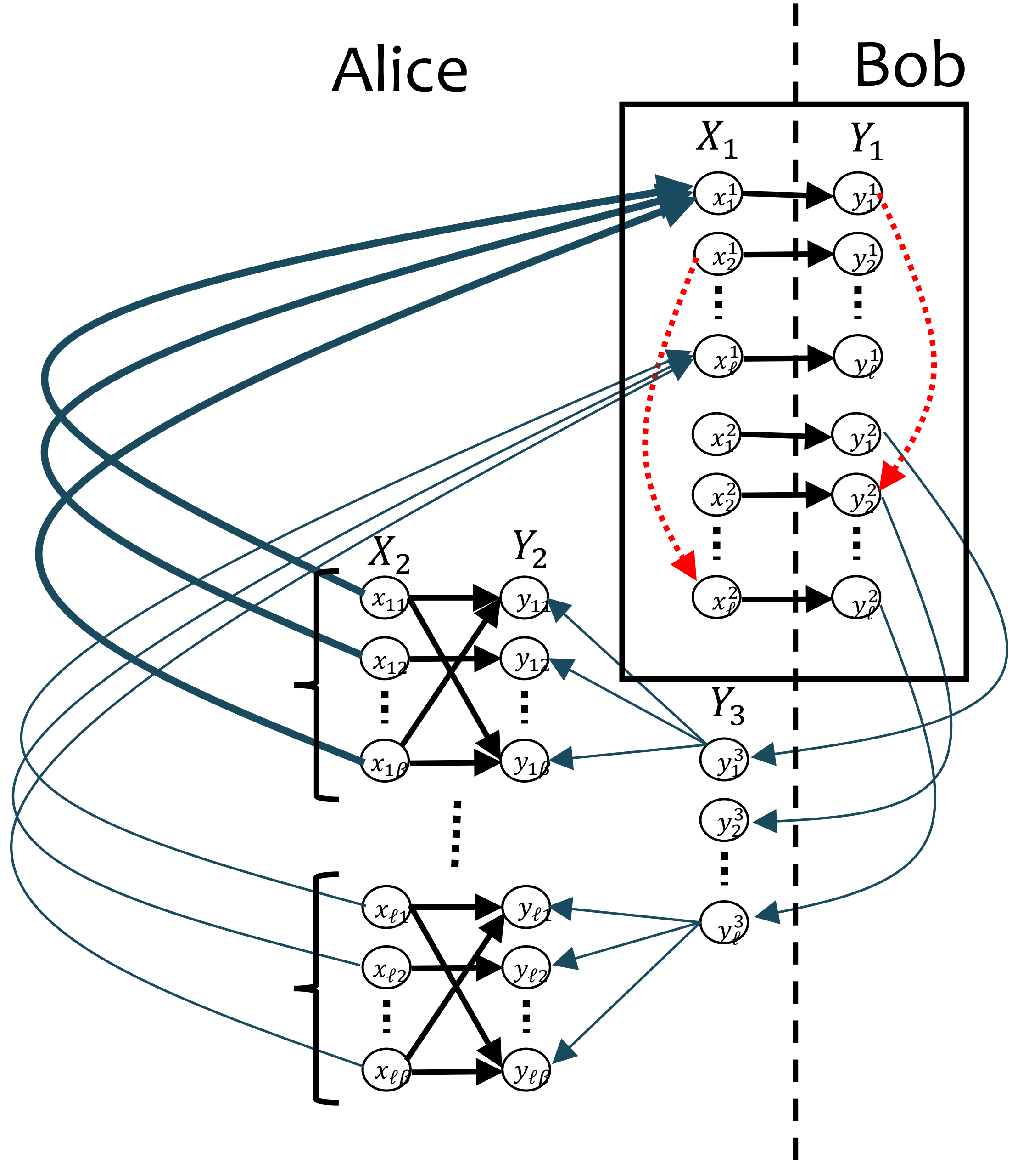
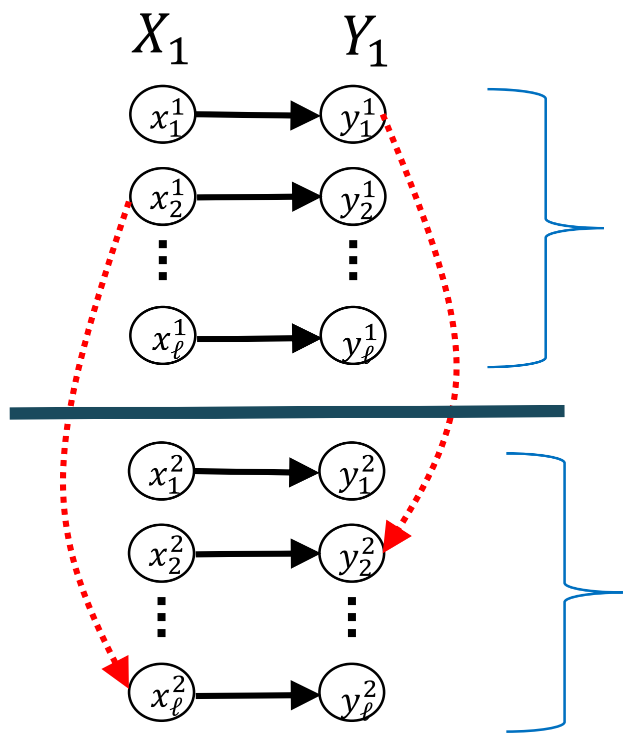


There are two inputs a, b of ℓ^2 bits a_{ij}, b_{ij} such that:

$$(x_i^1, x_j^2) \text{ is in } G \Leftrightarrow a_{ij} = 0$$

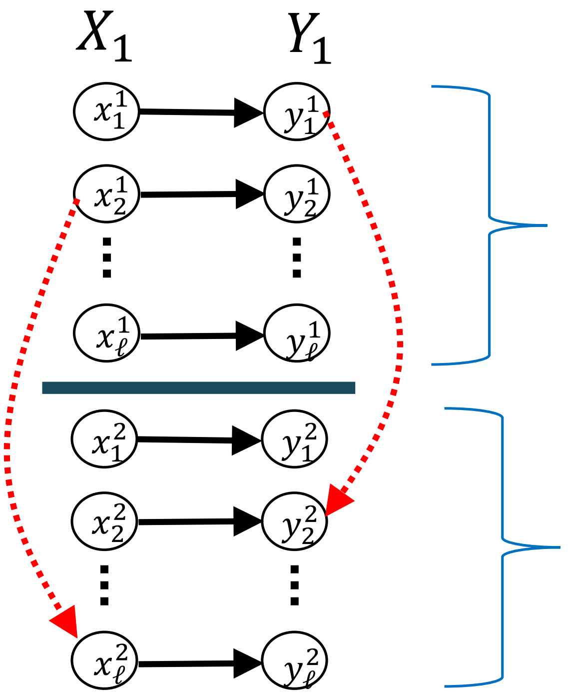
$$(y_i^1, y_j^2) \text{ is in } G \Leftrightarrow b_{ij} = 0$$



There is a directed path of length 2 from x_i^1 to y_j^2

$$\Leftrightarrow$$

$$a_{ij} = 0 \text{ or } b_{ij} = 0$$



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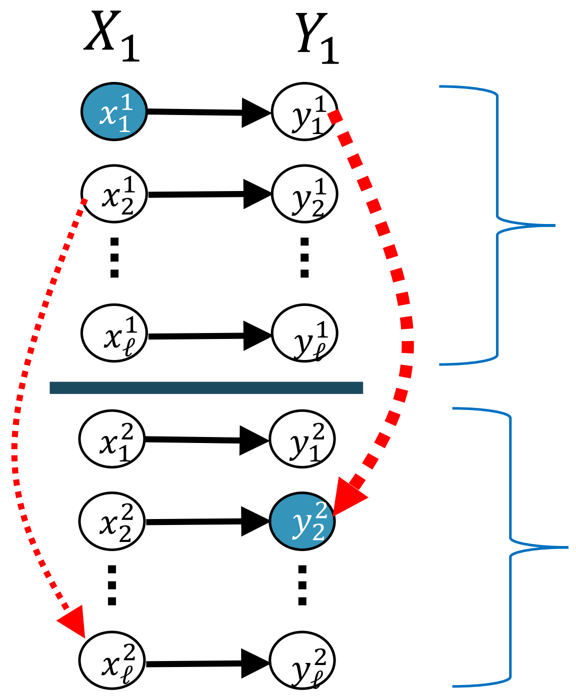

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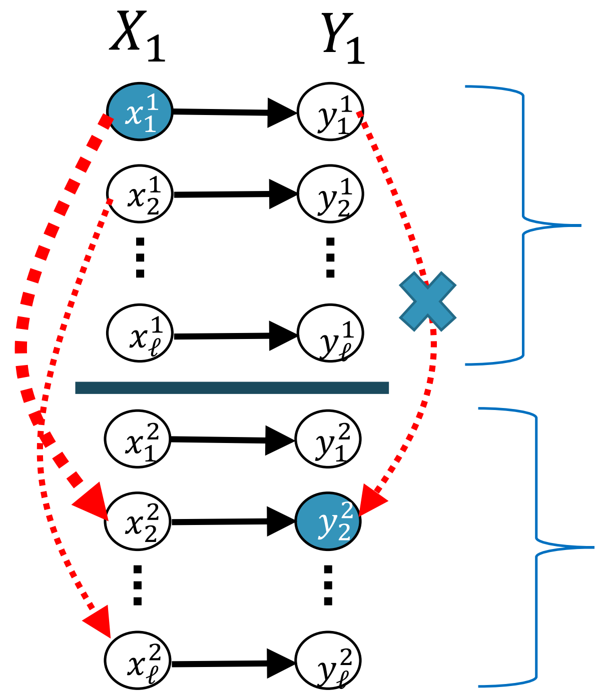

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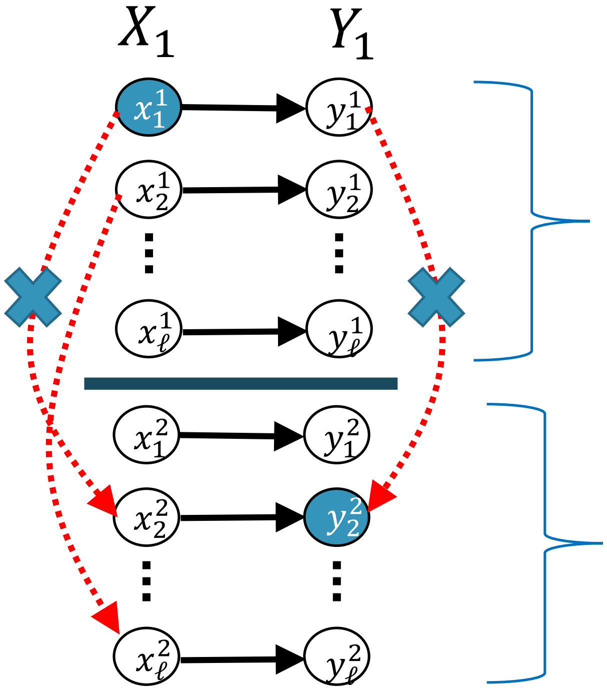
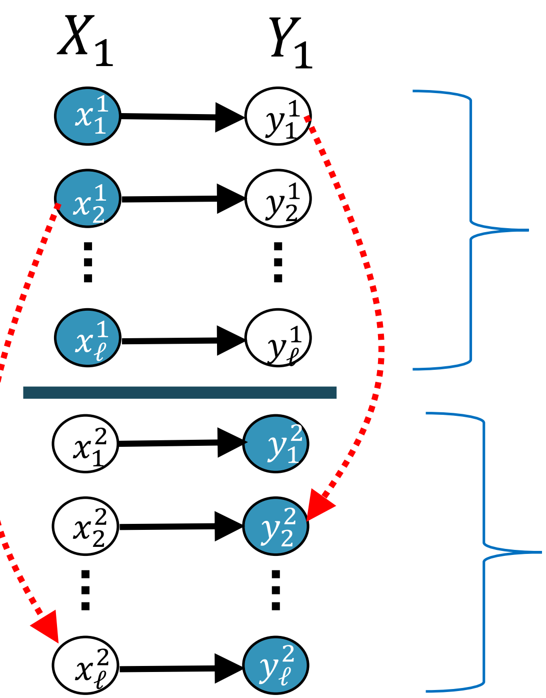


Conclusion:

a, b are *disjoint*

$\Leftrightarrow$

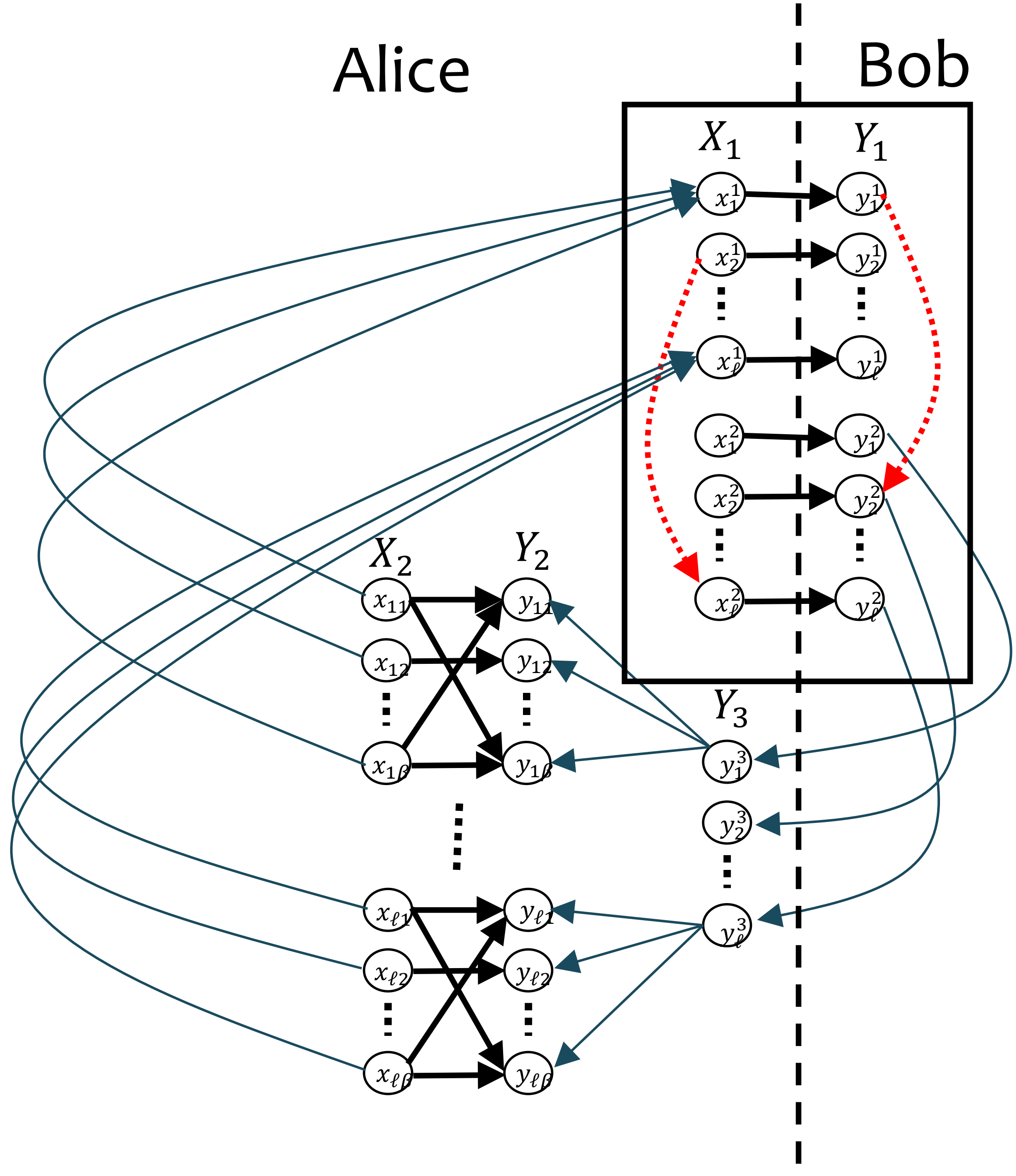
there is a path of length 2 from x_i^1 to y_j^2 for all i, j



G contains a sparse 5-spanner

$\Leftrightarrow$

a, b are disjoint



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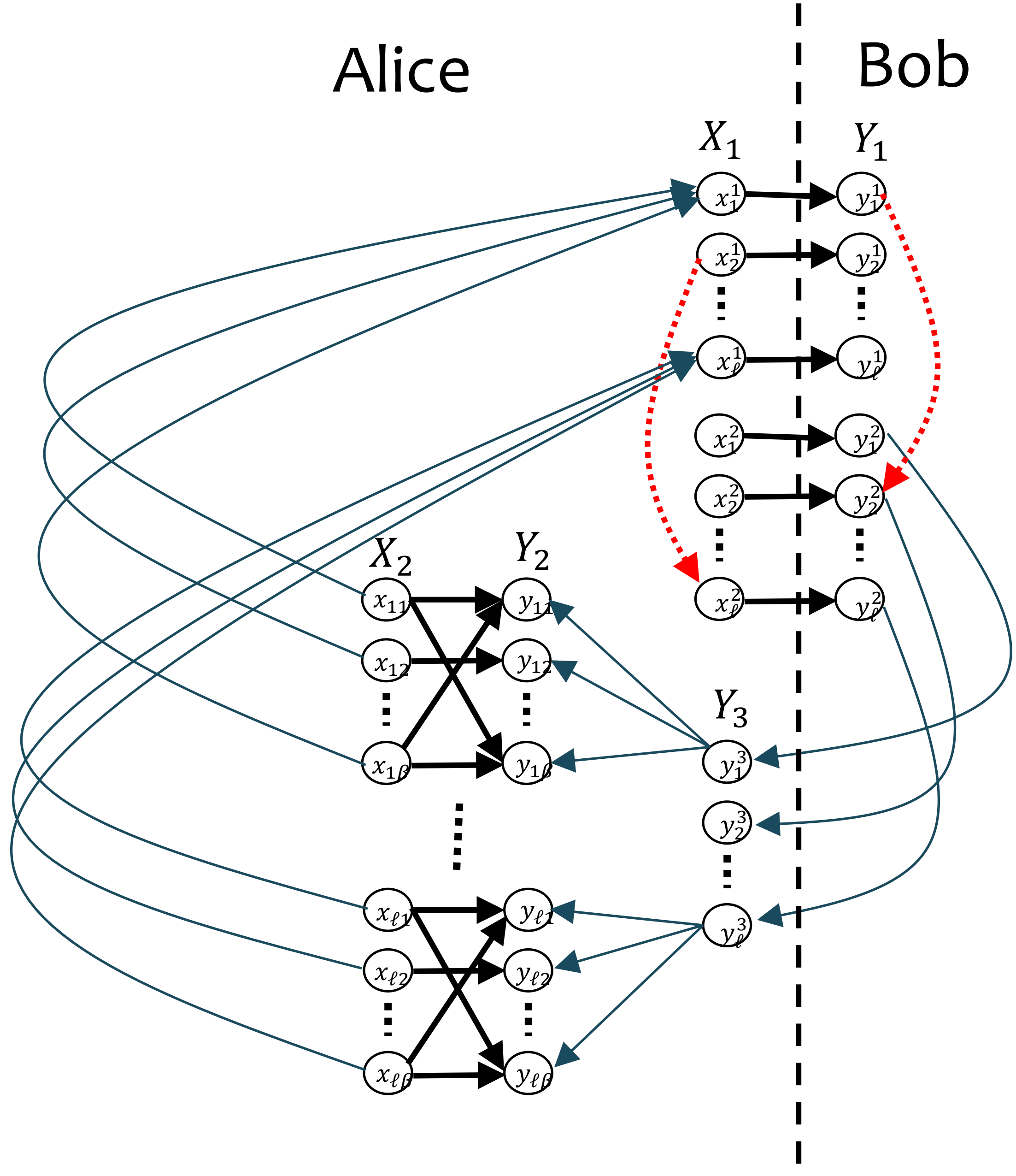

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$\Longleftrightarrow$
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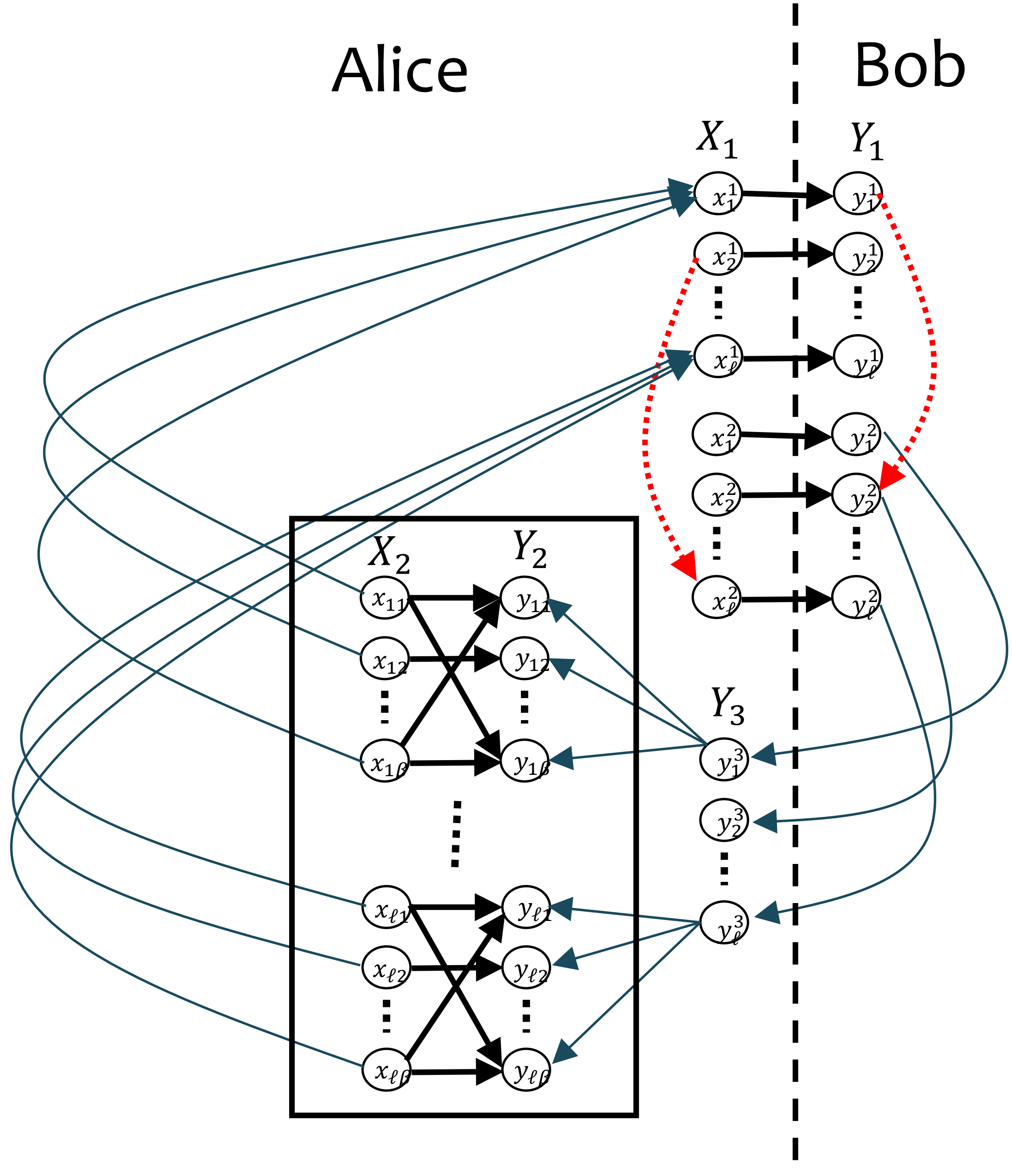

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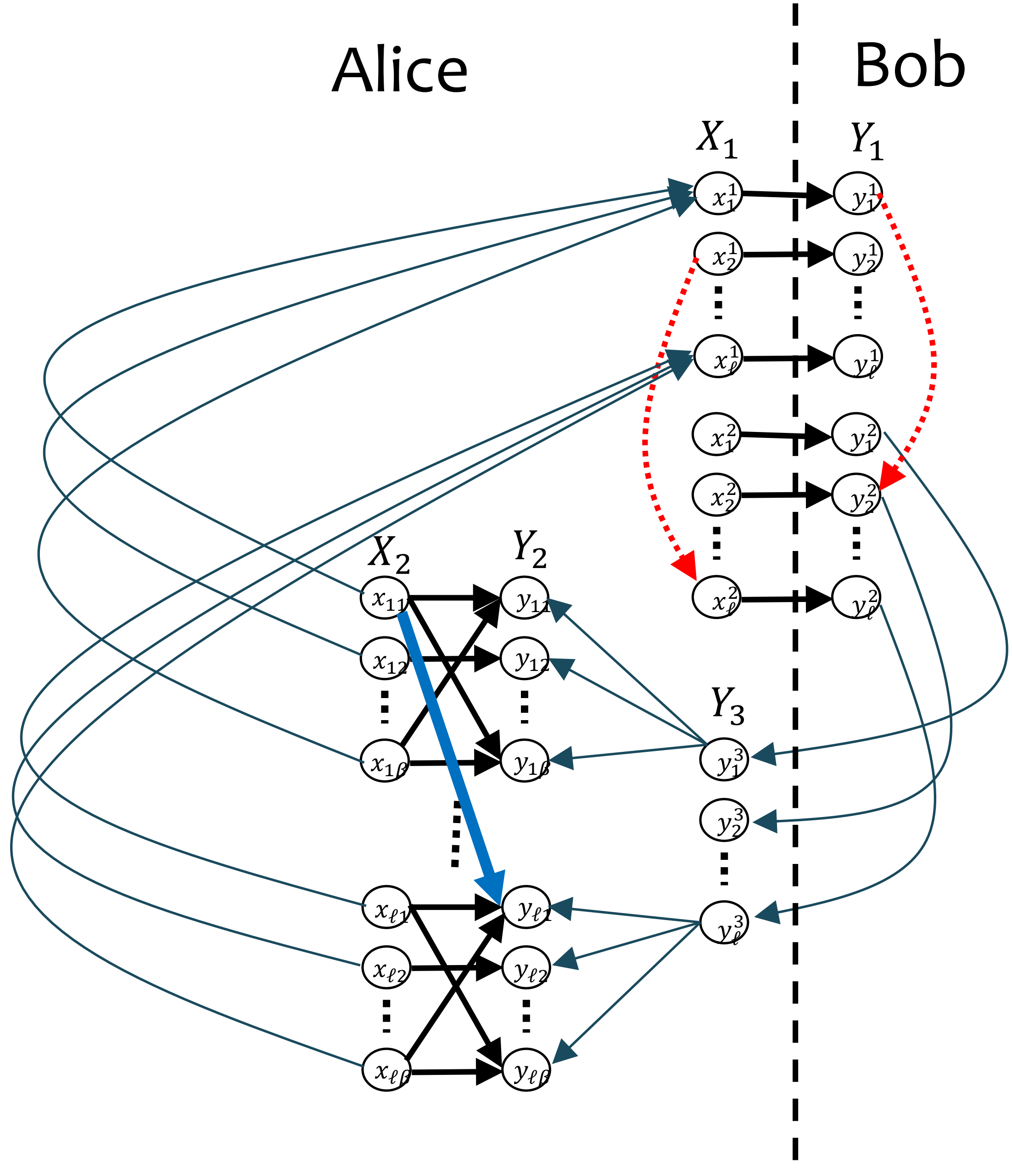

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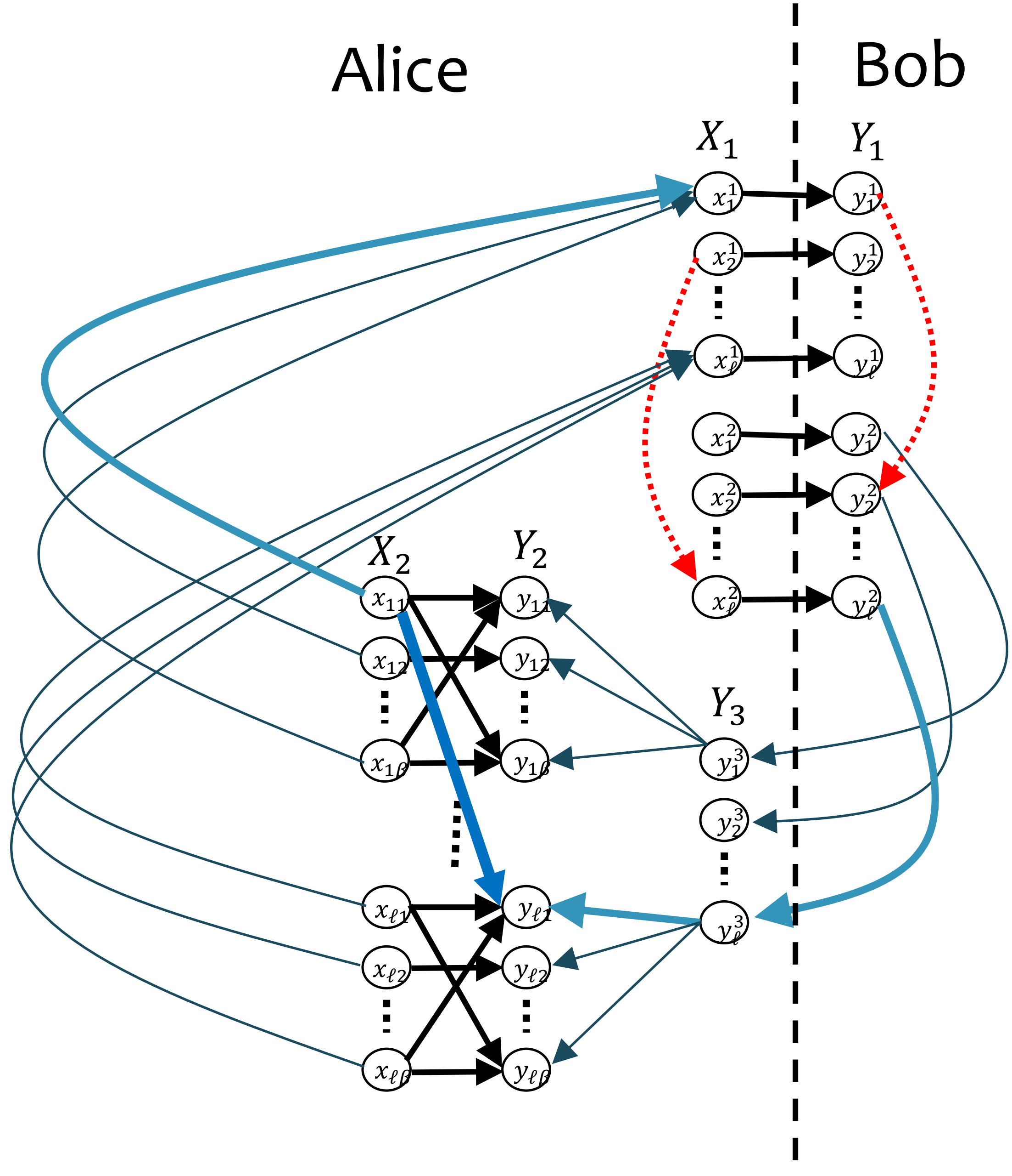

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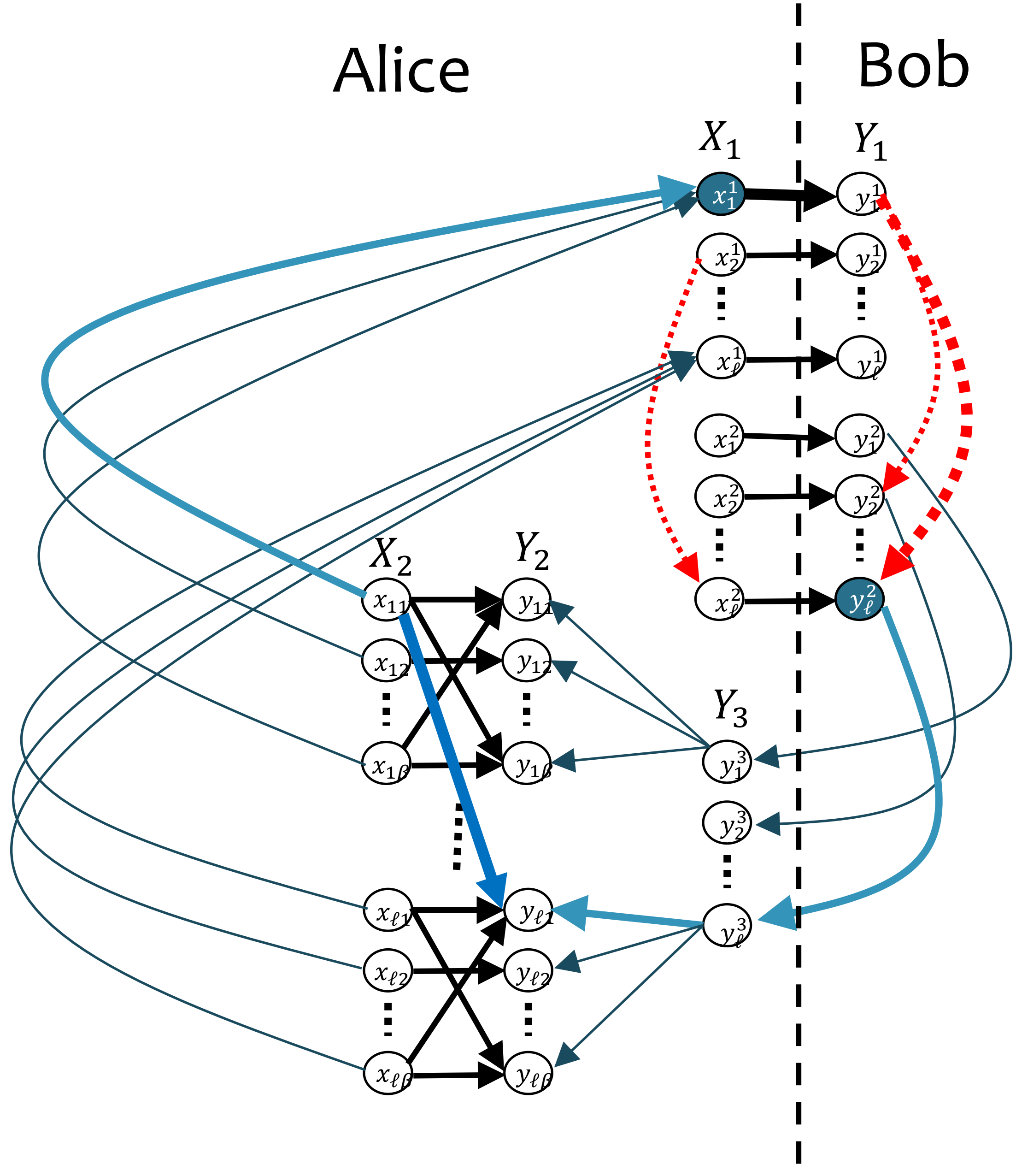


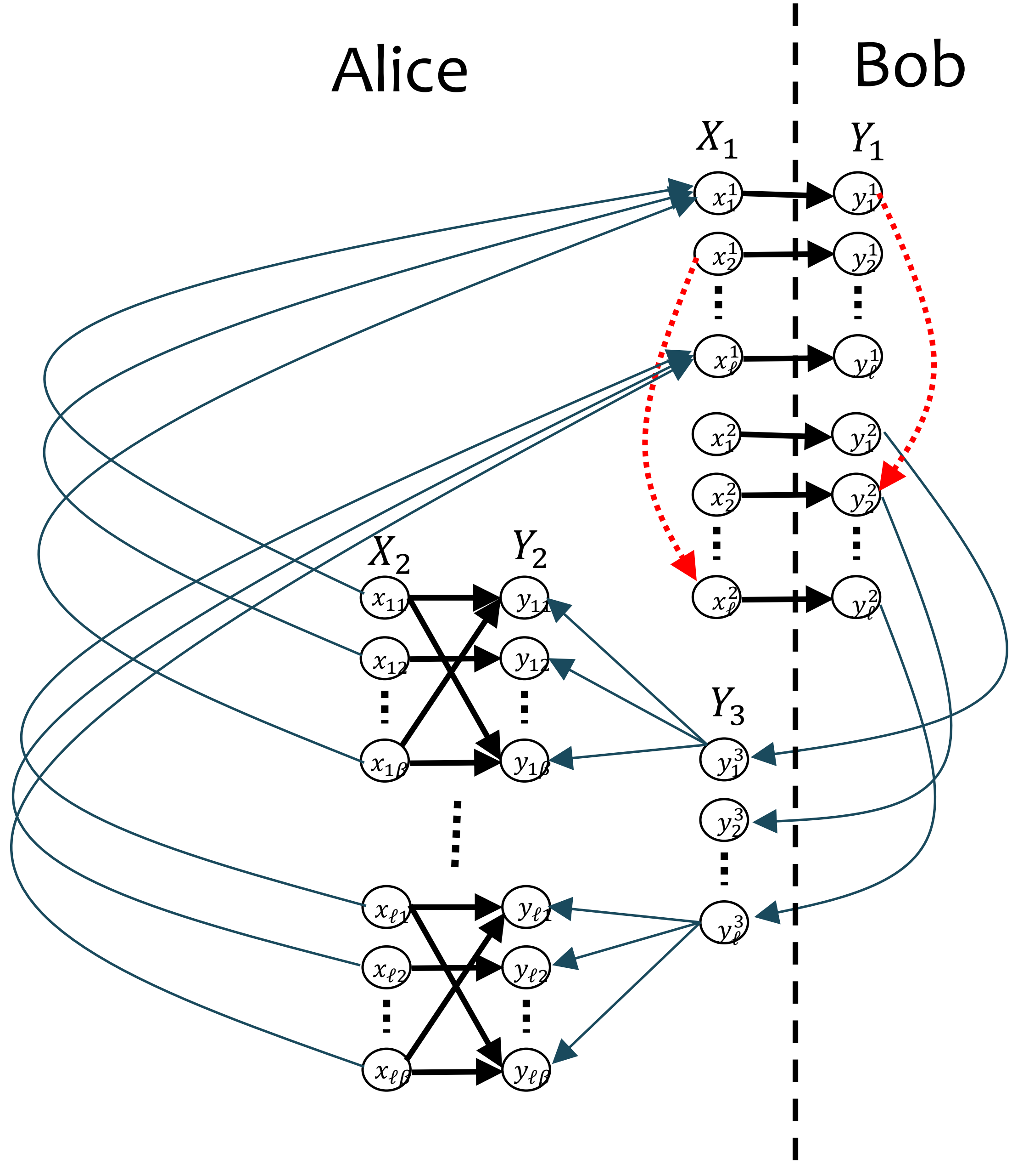
If a, b are **not** disjoint

There are i, j where there is **no** directed path of length 2 from

x_i^1 to y_j^2

We need to take at least $\boldsymbol{\beta^2}$ **edges** to the spanner

We choose $\beta \approx \sqrt{\alpha n}$



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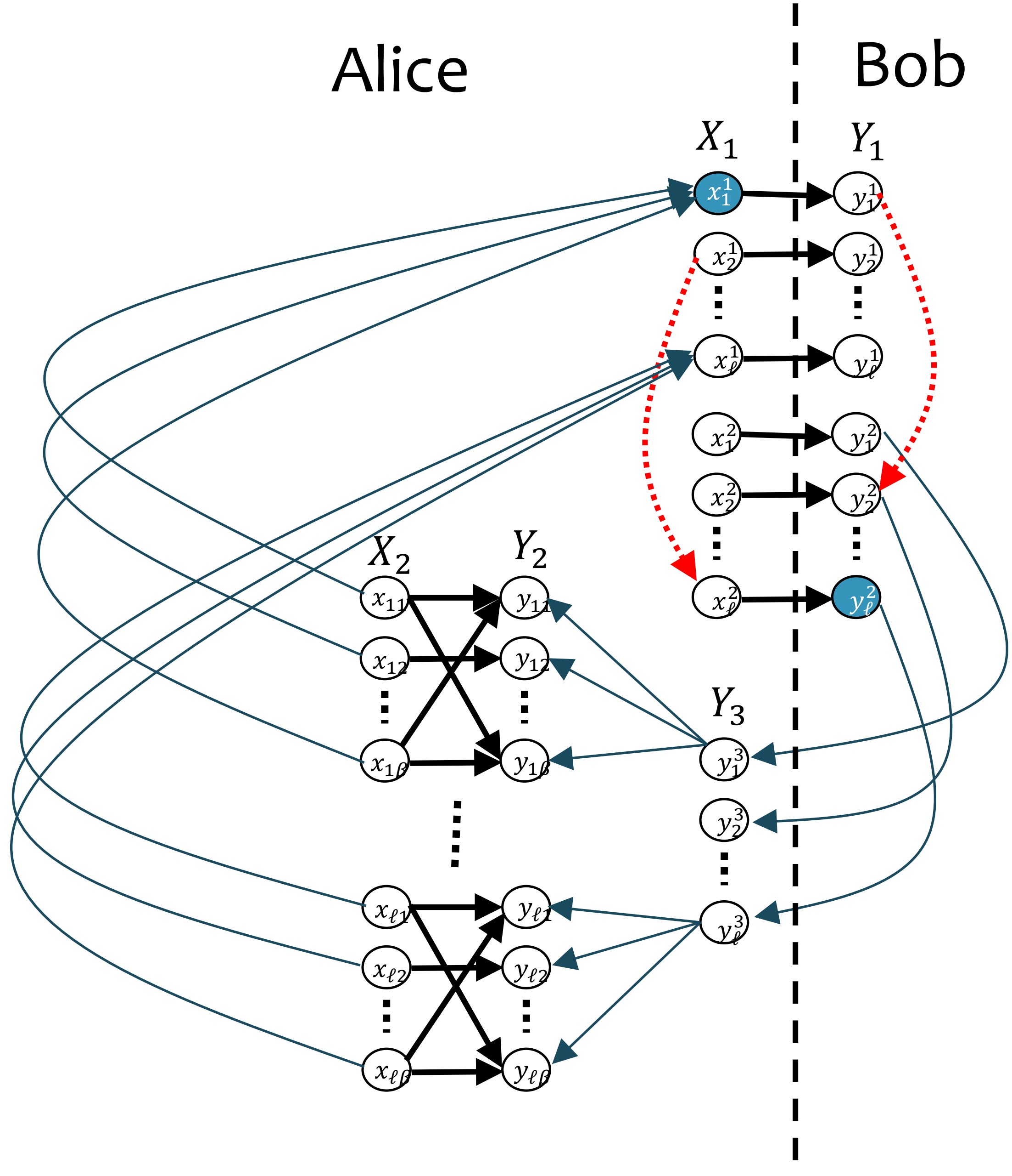

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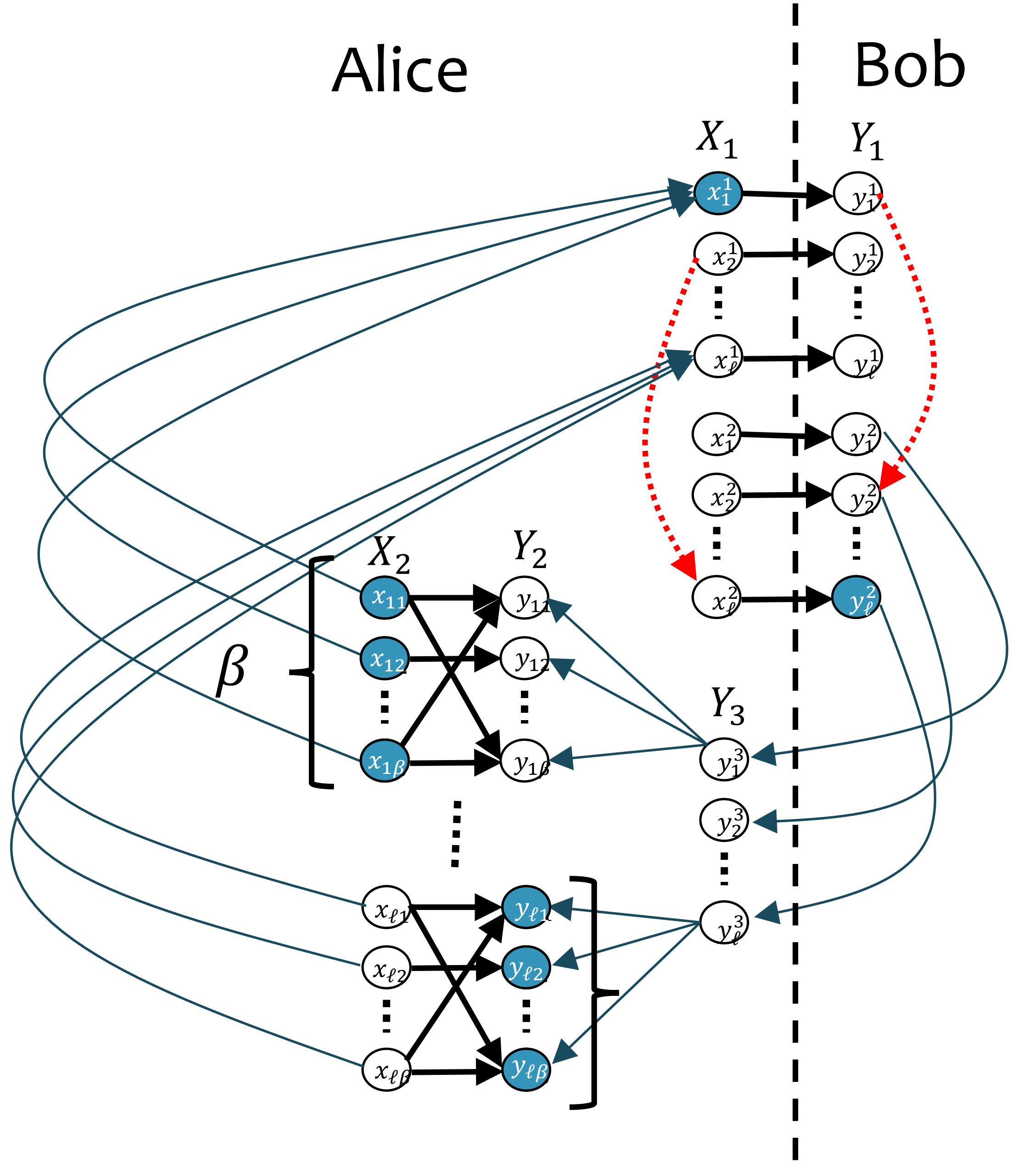

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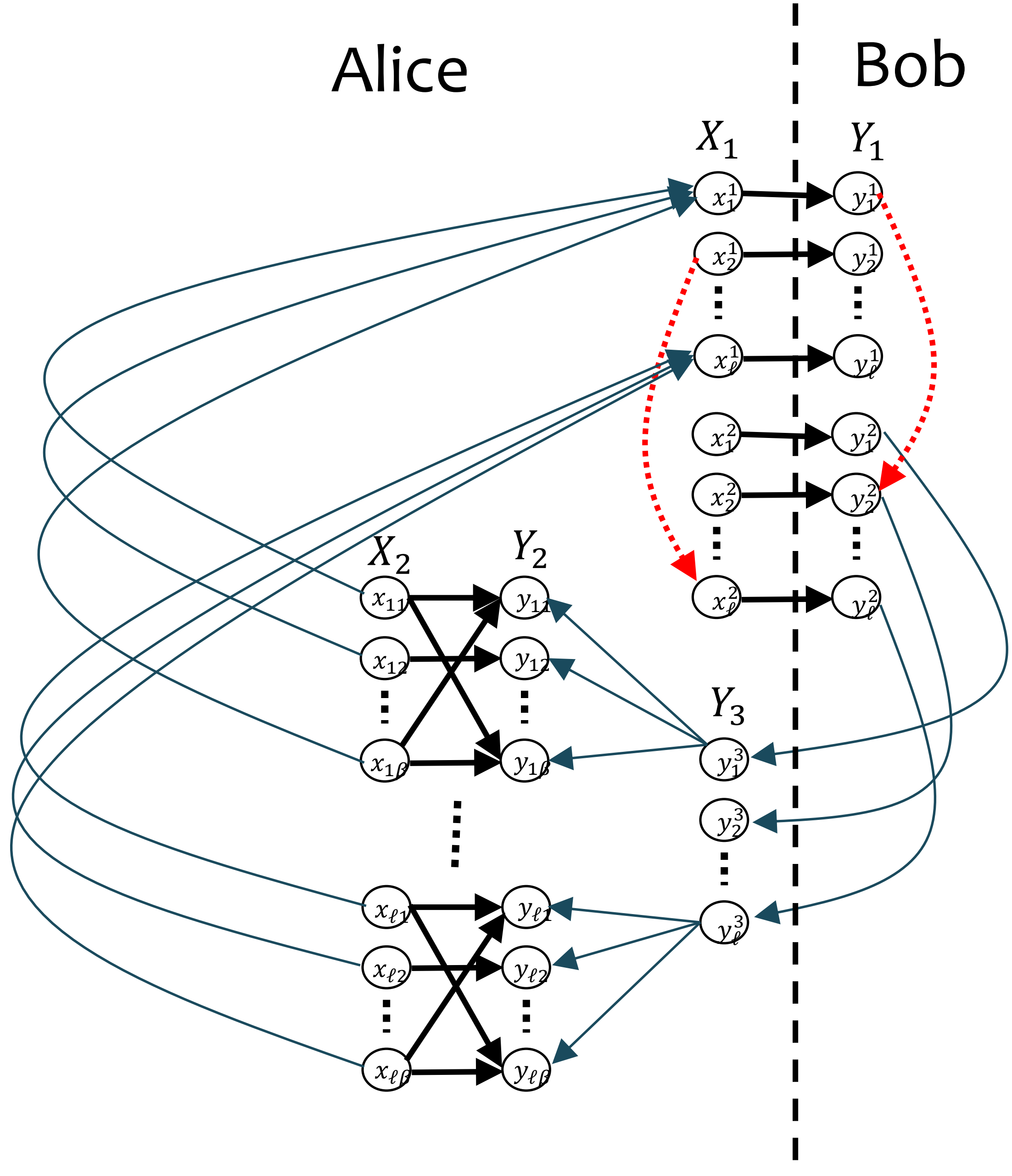


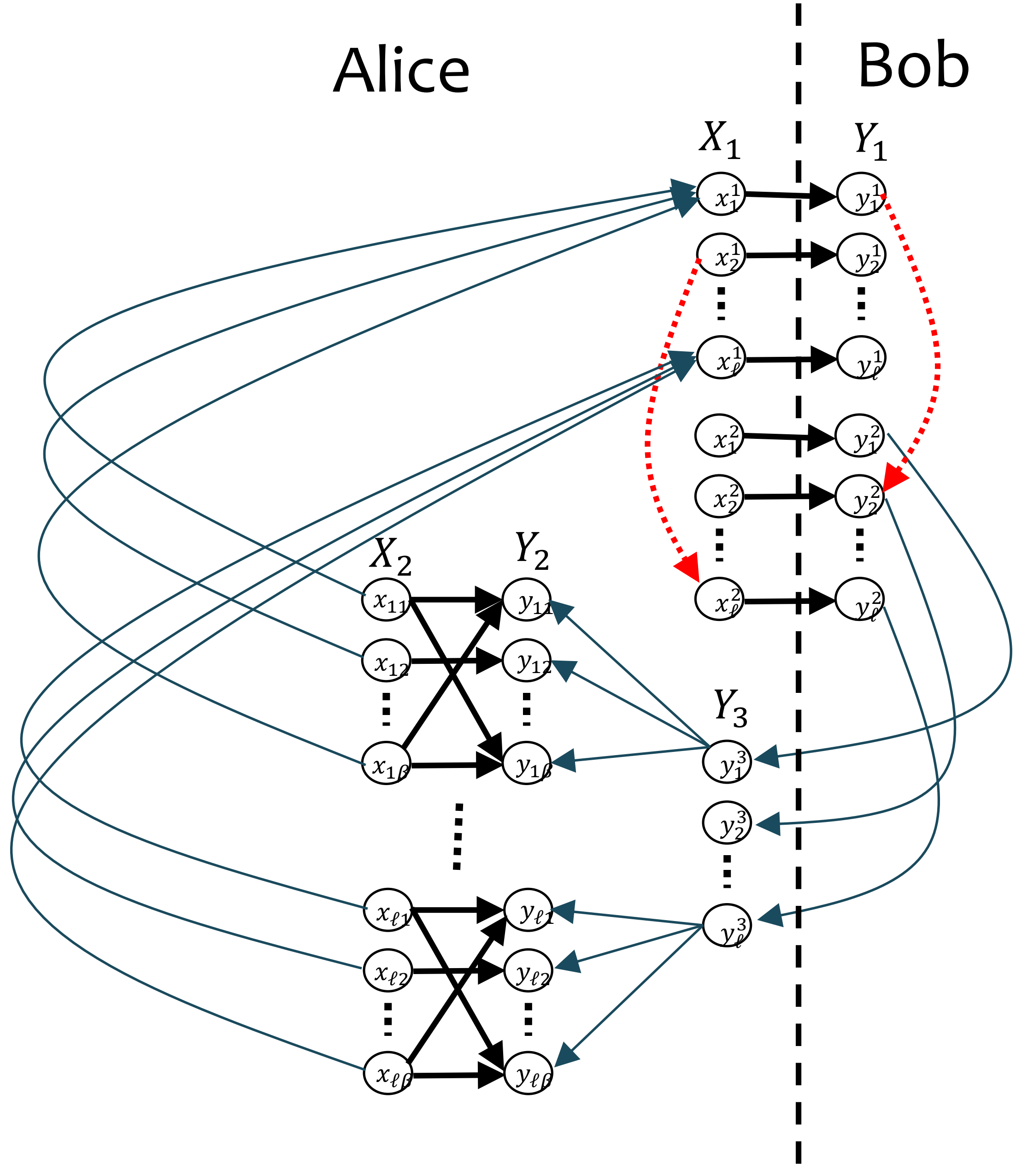
$\Omega(\ell^2)$ lower bound for set-disjointness

Approximation algorithm for 5-spanners in $O\big(T(n)\big)$ rounds

A protocol for disjointness that takes

$O(T(n) \log n\, |CUT_{A-B}|)$

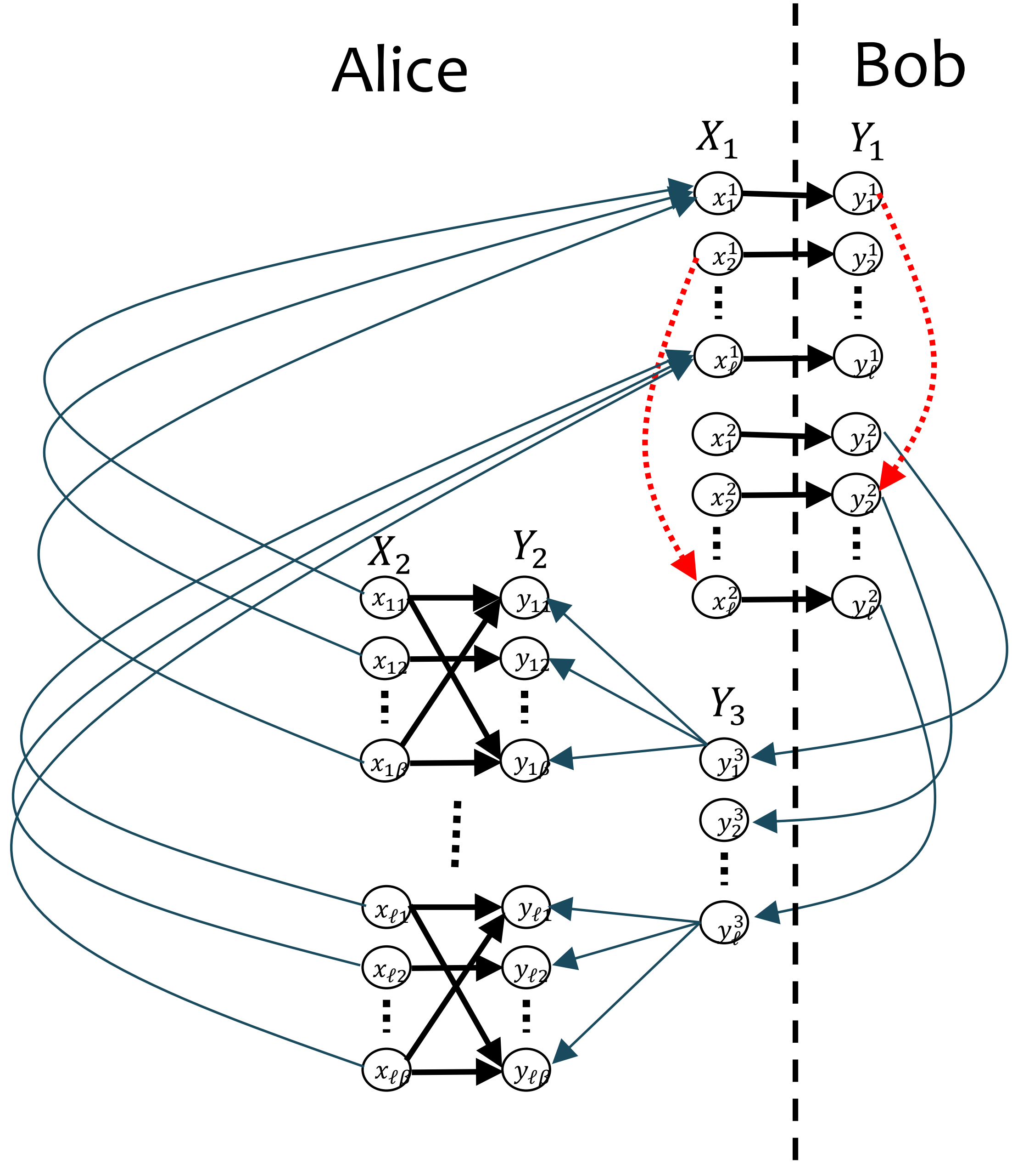
bits



$$|CUT_{A-B}| = \theta(\ell)$$

$$T(n) \cdot \log n \cdot \ell = \Omega(\ell^2)$$

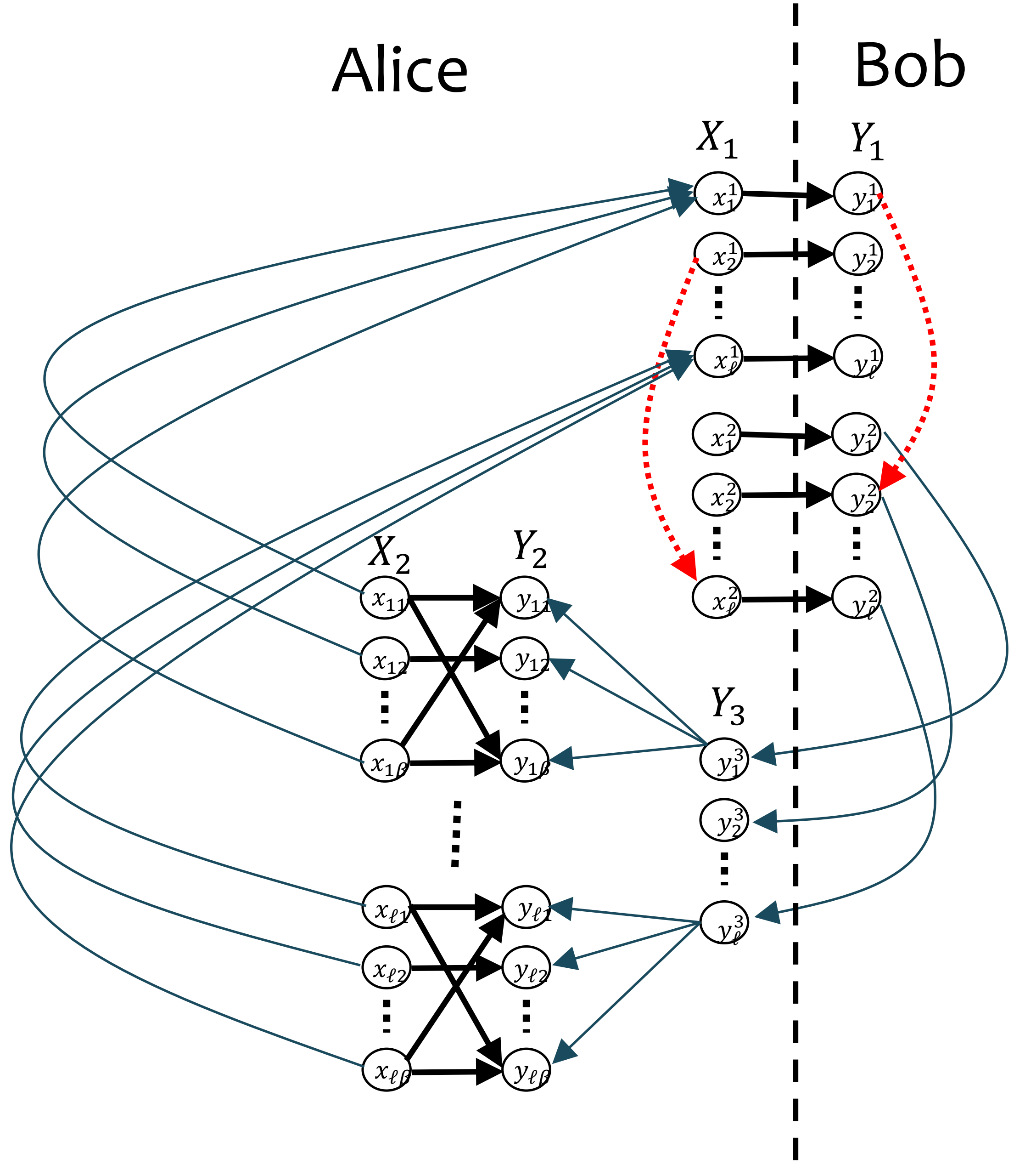
$$T(n) = \Omega\left(\frac{\ell}{\log n}\right)$$



$$T(n) = \Omega\left(\frac{\ell}{\log n}\right)$$

$$\beta \approx \sqrt{\alpha n}$$

$$\ell = \theta(n/\beta) = \theta(\sqrt{n/\alpha})$$



Conclusion

- $\widetilde{\Omega}(\sqrt{n/\alpha})$ rounds are required for constructing an α-approximation for directed 5-spanner (holds also for $k \geq 5$).

Other results:

- For **deterministic** algorithms, we can improve the lower bound to $\widetilde{\Omega}(n/\sqrt{\alpha})$ using gap-disjointness.
- For **weighted** graphs, we can improve the lower bound to $\widetilde{\Omega}(n)$ for directed graphs, and $\widetilde{\Omega}(n/k)$ for undirected graphs.

Other variants

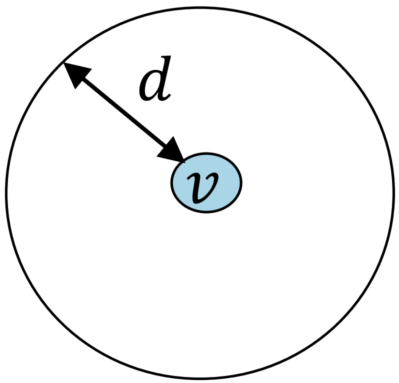
- We can use this construction to show hardness results for additional variants, such as the client-server variant.
- The main open question is the undirected unweighted case.

Algorithm in the LOCAL model

We can get $(1+\epsilon)$-approximation in $O(poly(\log n/\epsilon))$ rounds in the LOCAL model. [inspired by Ghaffari, Kuhn, and Maus, 2017]

A sequential algorithm

- $v_1, v_2, \dots, v_n$
- $B_d(v)$ = the ball of radius d around v
- $g(v, d)$ = the size of an **optimal $\boldsymbol{k}$-spanner** for the *uncovered* edges in $\boldsymbol{B_d(v)}$

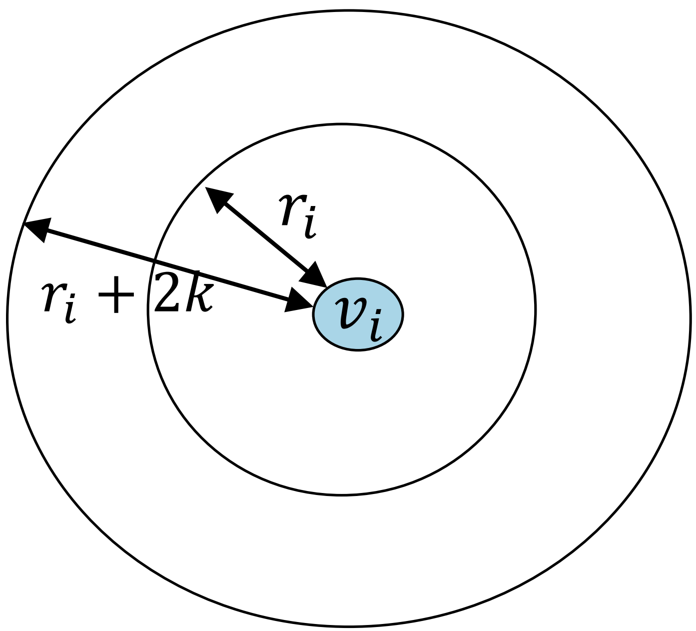


A sequential algorithm

$g(v, d)$ = the size of an **optimal $\boldsymbol{k}$-spanner** for $\boldsymbol{B_d(v)}$

In iteration i:

find r_i such that $g(v_i, r_i + 2k) \leq (1 + \epsilon) g(v_i, r_i)$



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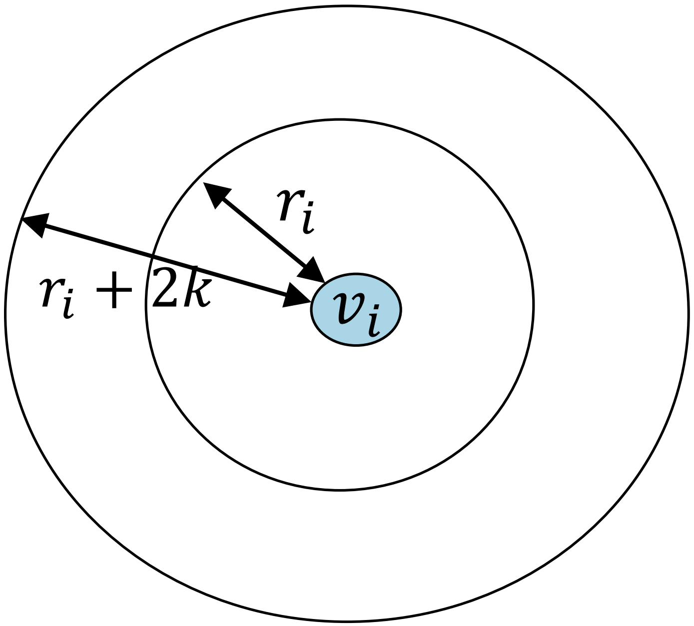
$g(v, d)$ = the size of an **optimal $\boldsymbol{k}$-spanner** for $\boldsymbol{B_d(v)}$

<u>In iteration i:</u>

find r_i such that $g(v_i, r_i + 2k) \leq (1 + \epsilon) g(v_i, r_i)$

$r_i = O(\log n / \epsilon)$

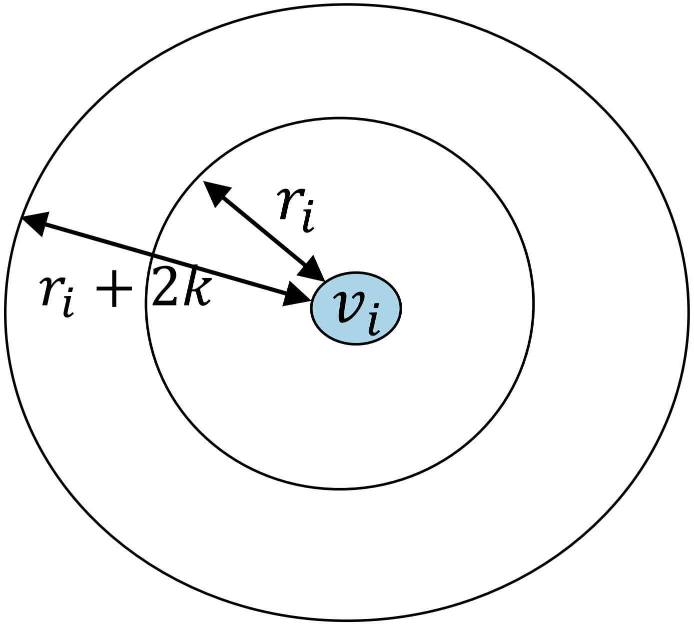


A sequential algorithm

In iteration i:

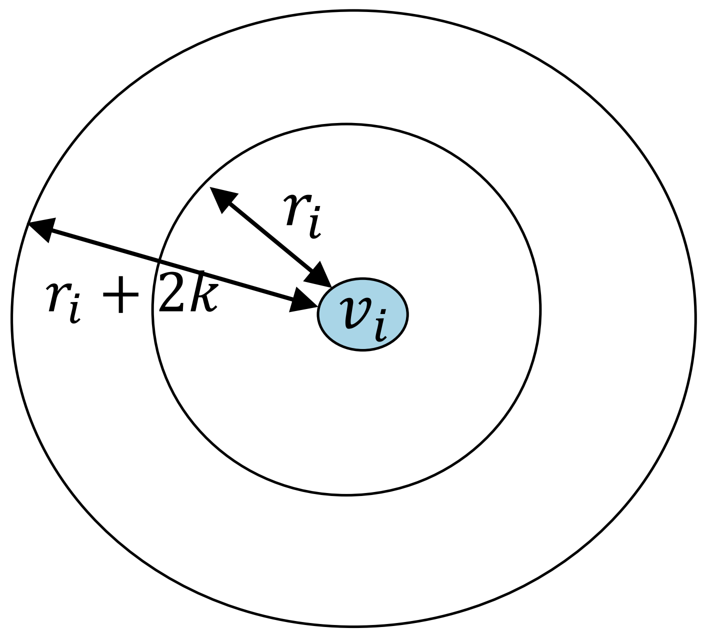
find r_i such that $g(v_i, r_i + 2k) \leq (1+\epsilon) g(v_i, r_i)$

add an **optimal spanner** for $\boldsymbol{B_{r_i+2k}(v)}$



Approximation ratio analysis

- H^* - an optimal spanner
- E_i $=$ the uncovered edges in $B_{r_i}(v_i)$ before iteration i
- $H_i^* \subseteq H^*$ = minimum k-spanner for E_i



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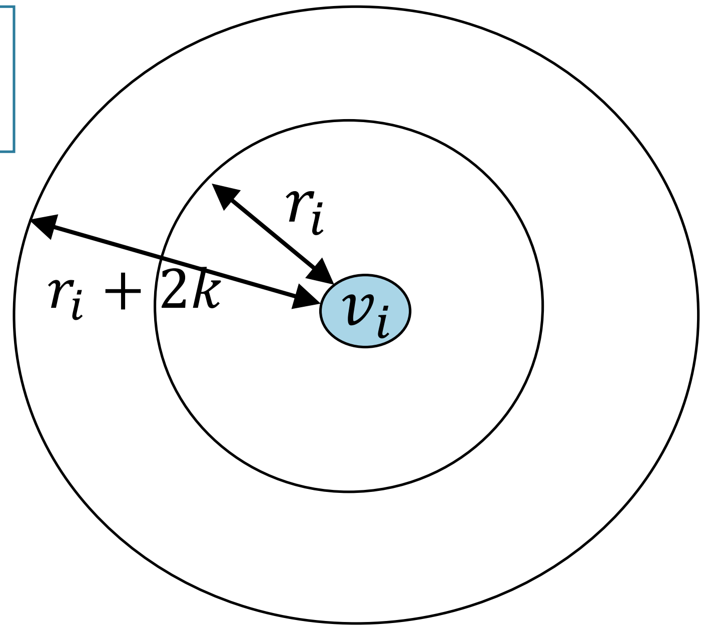
- E_i = the uncovered edges in $B_{r_i}(v_i)$ before iteration i
- $H_i^* \subseteq H^*$ = minimum k-spanner for E_i

E_i, E_j are at distance at least $2k+1$

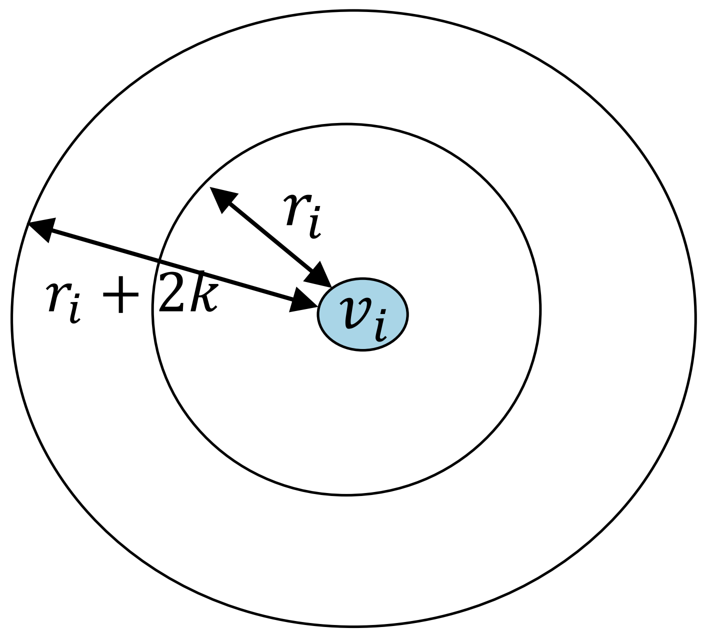
The subsets H_i^* are disjoint



Approximation ratio analysis

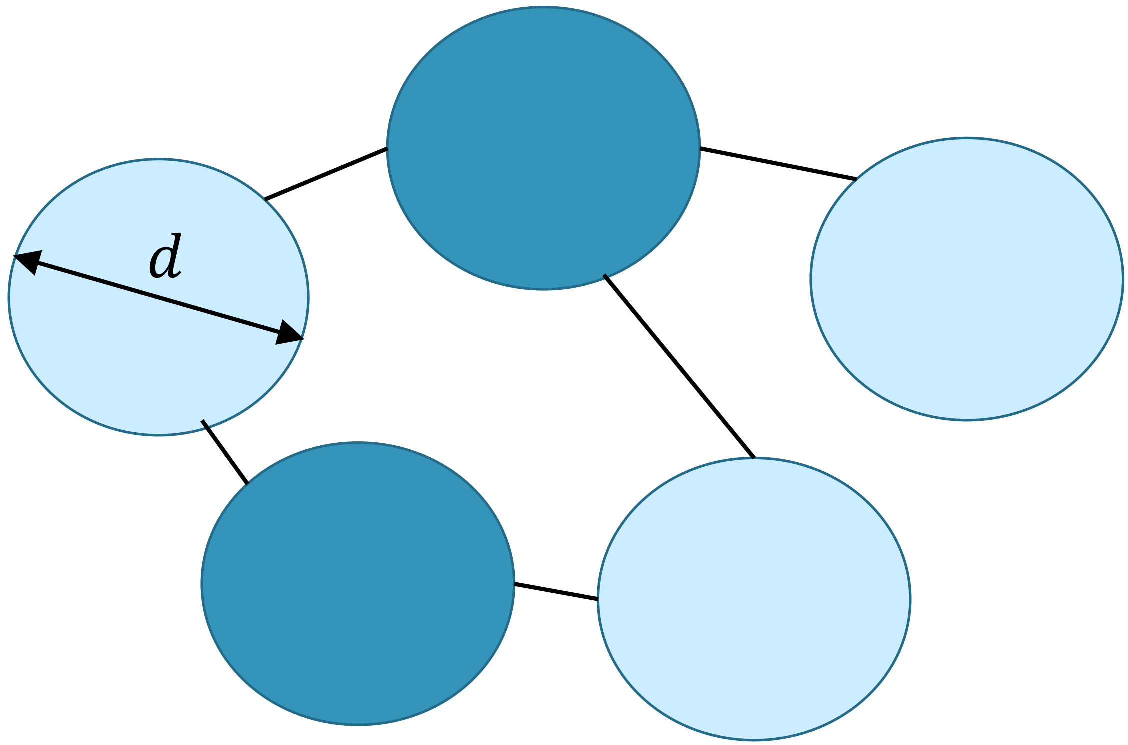
The number of edges in our spanner is at most

$$\sum_{i=1}^{n}(1+\epsilon)\,|H_i^*| \leq (1+\epsilon)|H^*|$$



Distributed algorithm

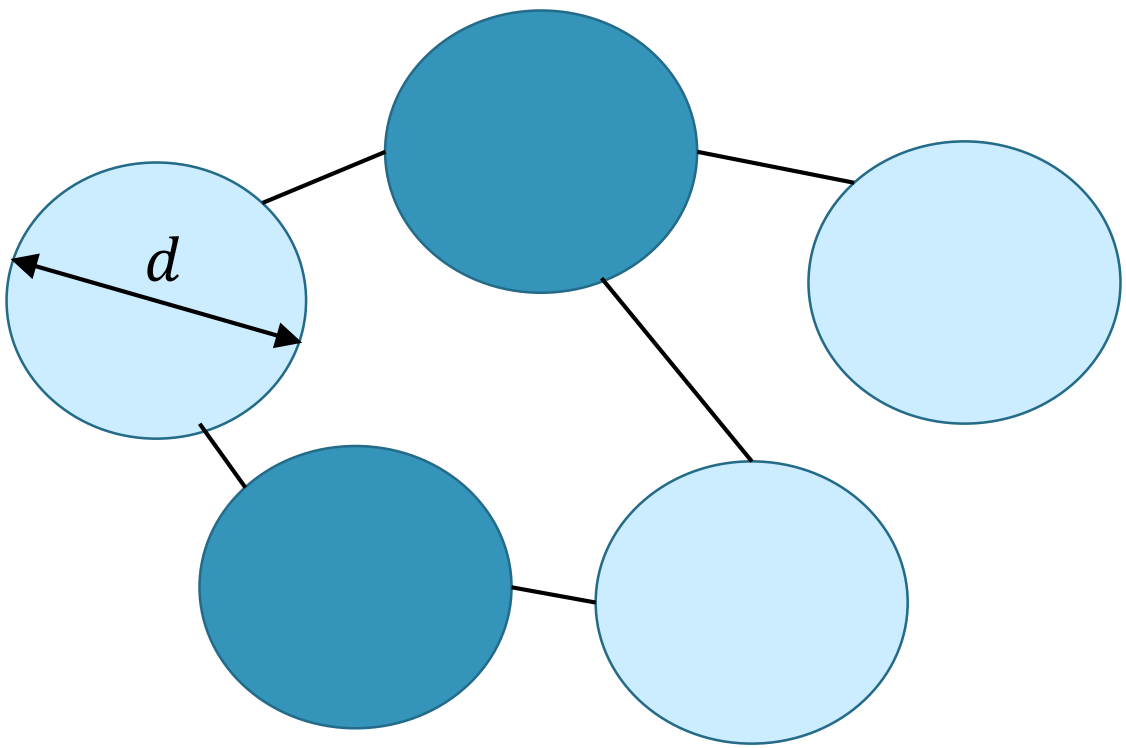
We use (d, c)-**network decomposition**, with $d = c = O(\log n)$
[Linial and Saks, 1993]



Distributed algorithm

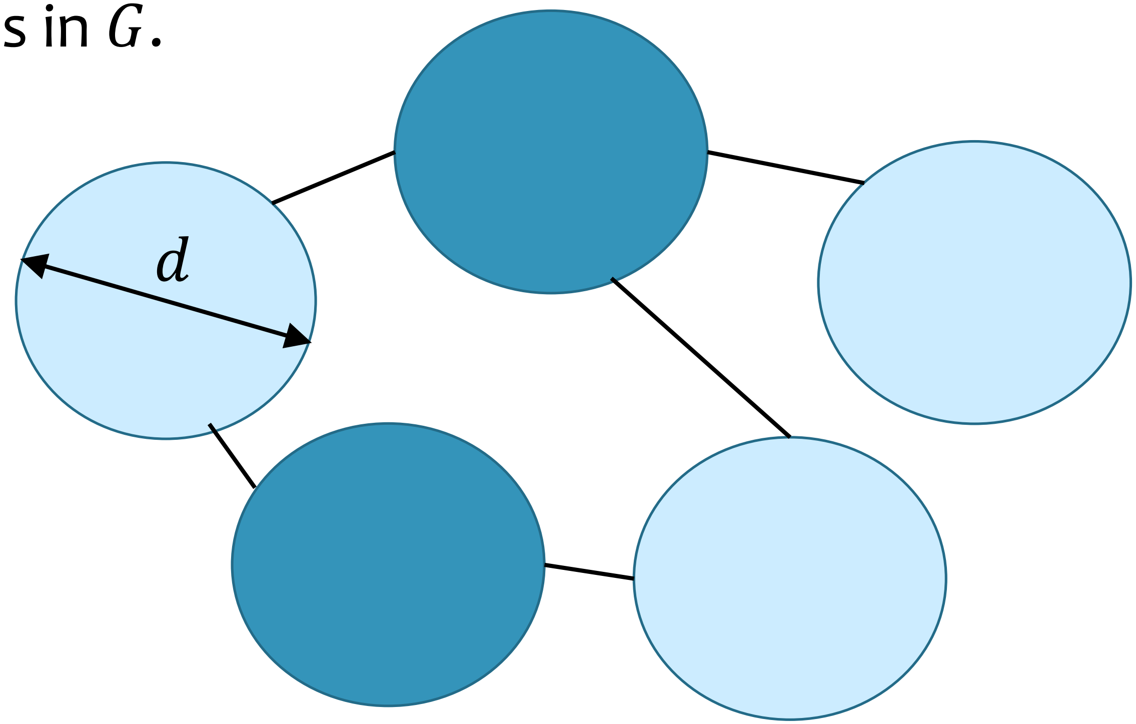
- We choose $r = O(\log n / \epsilon)$ such that $r > r_i + 4k$ for all i
- We compute a **network decomposition of G^r**

($O(poly(\log n / \epsilon))$ rounds in the LOCAL model).



Distributed algorithm

- The label of a vertex v is (col_v, id_v) → **order** of the vertices
- We simulate the sequential algorithm according to **increasing order of the colors**.
- The computations depend only on the ***r*-neighborhood** of vertices in G.



Conclusion

- We can get $(1+\epsilon)$-approximation in $O(poly(\log n/\epsilon))$ rounds in the LOCAL model.
- This algorithm is based on learning neighborhoods of polylogarithmic size and solving NP-complete problems.

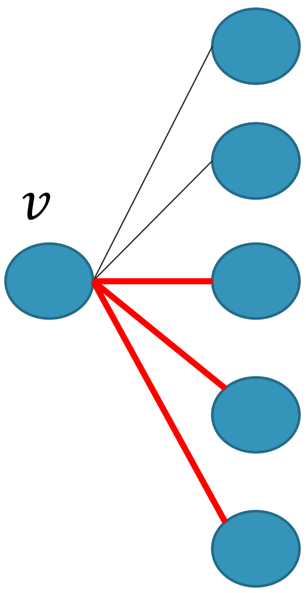
2-spanners

- There is an $O(\log n)$-round $O(\log n)$-approximation in expectation using only polynomial local computations [Dinitz and Krauthgamer, 2011]

- Can we give an $O\left(\log \frac{|E|}{|V|}\right)$-approximation?
- Can we guarantee the approximation ratio?
- Can we give an algorithm in the CONGEST model?
- What about lower bounds?

Stars and Densities

- A ***star*** around a vertex v, is a subset $\boldsymbol{S}$ of edges between v to some of its neighbors.
- The ***density*** of a star S is $\frac{C_S}{|S|}$ where C_S is the number of edges *covered* by the star S.



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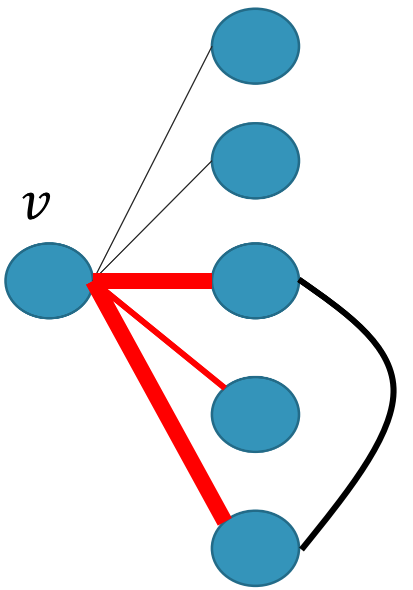

Sequential Greedy Algorithm

[Kortsarz and Peleg 1994]

- At each step, find the **densest star** in the graph, and add its edges to the spanner.
- Continue until all edges are covered.

- Achieves approximation ratio of $O\left(\log \frac{|E|}{|V|}\right)$

Distributed Algorithm – take 1

- At each step, find the **densest star** in the graph, and add its edges to the spanner.
- Continue until all edges are covered.

Distributed Algorithm – take 2

- At each step, find all the stars that are densest in their **local 2-neighborhood**, and add their edges to the spanner.
- Continue until all edges are covered.

Distributed Algorithm – take 3

- At each step, find all the stars that are densest in their local 2-neighborhood, they are the ***candidates***.
- Each candidate chooses a random number $r \in [0,1]$.
- Each uncovered edge **votes** to the first candidate that covers it.
- A star is added to the spanner if it gets **at least $\frac{1}{8}$ of the votes** of the edges it covers.

Distributed Algorithm – take 3

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Approximation ratio

- For $e \in E$, we define $cost(e) = \frac{1}{\rho}$ if e is covered by a star it votes for and has density ρ, and $cost(e) = 0$ otherwise.
- We show:

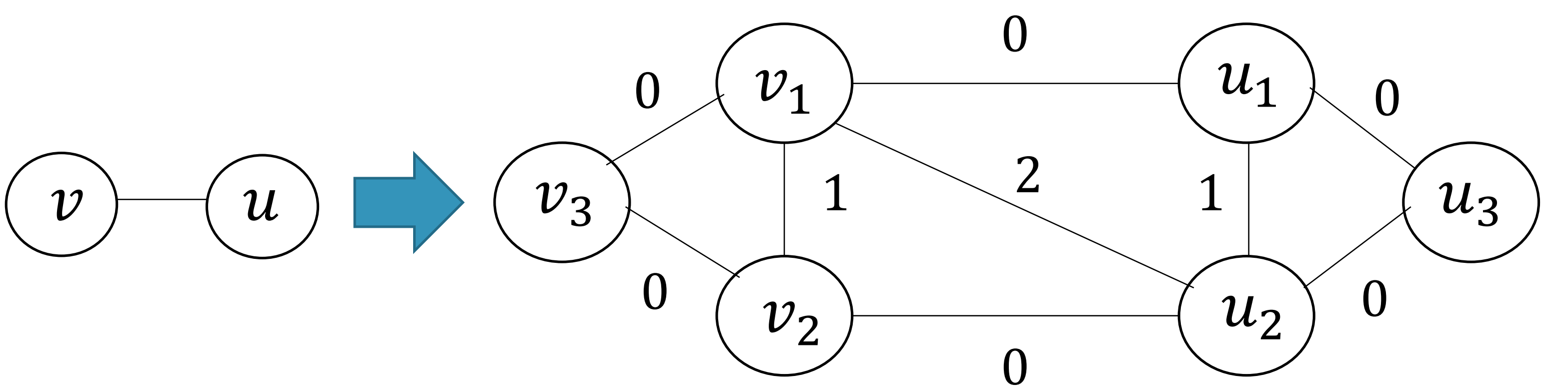
$$|H| \leq 8 \sum_{e \in E} cost(e) \leq O\left(\log \frac{|E|}{|V|}\right) |H^*|$$

Conclusion

- We can show that this approach guarantees an approximation of $O\left(\log \frac{|E|}{|V|}\right)$ in $O(\log n \log \Delta)$ rounds w.h.p.
- Extends also to the weighted, directed and client-server variants.
- We can also show that $\Omega\left(\frac{\log \Delta}{\log \log \Delta}\right)$ or $\Omega\left(\sqrt{\frac{\log n}{\log \log n}}\right)$ rounds are required for a logarithmic approximation for weighted 2-spanner. [reduction from Kuhn, Moscibroda, and Wattenhofer, 2016]

Lower bound for weighted 2-spanner

- We show a reduction from vertex cover



Open questions

Hardness of Approximation:

- Is it possible to show **separations** between the **LOCAL** and **CONGEST** models for other problems?
- **Undirected unweighted** k-spanner

2-spanner:

- Is it possible to show a **CONGEST** algorithm?